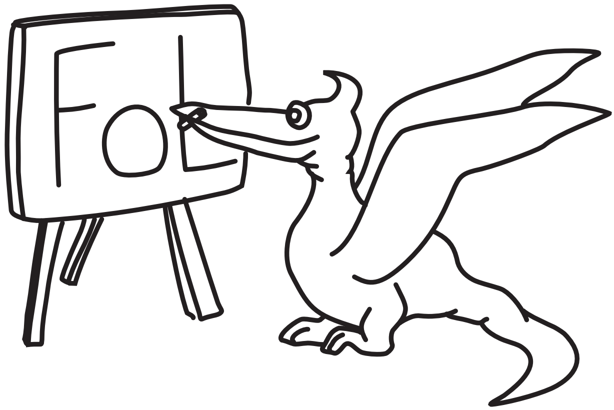
Solutions of 11th Physics Brawl Online



Problem 1 ... an unfortunate trolleybus

3 points

A trolleybus of total mass $M_\mathrm{t} = 15\,\mathrm{tonnes}$ *(without any passengers) and of volume* $V_\mathrm{t} = 90\,\mathrm{m^3}$ *is entering a bridge over a river. Unfortunately, the bridge is being reconstructed, and careless workers forgot to put up no entry signs. The trolleybus falls into the river. What is the maximum percentage of the trolleybus volume that can be occupied by passengers such that after the fall, the trolleybus ends up floating to the surface? Consider the density of the human body to be the same as the density of water. The trolleybus is airtight.*

Verča heard that there was a demand for a trolleybus problem.

Let us denote the sought ratio as x. The final mass of trolleybus loaded with passengers can be expressed as $M_\mathrm{t} + x\rho_\mathrm{w} V_\mathrm{t}$. If the trolleybus ought to float, the buoyancy force acting on a fully submerged trolleybus has to be at least as great as the gravity. Thus, we get the equation

$$(M_\mathrm{t} + x\rho_\mathrm{w} V_\mathrm{t})\, g = V_\mathrm{t}\rho_\mathrm{w} g\,.$$

From the expression above, we can easily express the ratio

$$x = \frac{V_\mathrm{t}\rho_\mathrm{w} - M_\mathrm{t}}{V_\mathrm{t}\rho_\mathrm{w}} = 1 - \frac{M_\mathrm{t}}{V_\mathrm{t}\rho_\mathrm{w}}\,.$$

After substituing the numbers, we get $x \doteq 83.3\,\%$.

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Problem 2 ... our antique clock

3 points

An antique pendulum clock has to be wound up every Sunday at the same time to show the correct time during the whole week. While winding up, one has to raise a weight $m = 5.6\,\mathrm{kg}$ *by* $h = 31\,\mathrm{cm}$ *to enable the clockwork to drive the pendulum. The pendulum is of length* $l = 64\,\mathrm{cm}$ *and almost all its mass is located at the lower end. How much energy (on average) is dissipated during one swing?*

Jarda is late sometimes.

The pendulum in time $t = 1\,\mathrm{week} = 604\,800\,\mathrm{s}$ "use up" potential energy of the weight, which is

$$E_\mathrm{p} = mgh \doteq 17.0\,\mathrm{J}\,,$$

therefore, the dissipation power is $P = \frac{E_\mathrm{p}}{t} \doteq 28.1\,\mathrm{\mu W}$. Regarding the problem description, we assume the pendulum to be mathematical

$$T = 2\pi\sqrt{\frac{l}{g}} \doteq 1.60\,\mathrm{s}\,.$$

Thus, the disipated energy during one swing is

$$E = P\frac{T}{2} = \frac{\pi m h}{t}\sqrt{lg} \doteq 22.6\,\mathrm{\mu J}\,.$$

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Problem 3 ... flywheel

3 points

Let us assume that we have a homogeneous disc (flywheel) of a mass $M = 10m$ and radius R. There is a fly called Ľubo of a mass m sitting on the flywheel at a distance $0.75R$ from the center of the disc, while the disc (together with Ľubo) is rotating with an angular frequency $\omega_0 = 1.20\,\mathrm{rad{\cdot}s^{-1}}$. Ľubo's friend Slavo, who has the same mass as Ľubo, suddenly sits at the edge of the disc, while he had zero angular momentum with respect to the axis of rotation of the disc. Assume that both flies are point masses and that the system (the flywheel + flies) is not subjected to any external torques. What is the angular frequency of the disc after Slavo sits on it?

Flies had been flying around Tomáš's head.

Let denote the numerical factors as $\alpha = 10$ and $\beta = 0.75$.

The moment of inertia of the disc is $I = \frac{1}{2}MR^2$. In our special case, the initial moment of inertia includes Ľubo, and thus

$$I_0 = \frac{1}{2}MR^2 + I_\mathrm{L} = \frac{1}{2}\alpha mR^2 + \beta^2 mR^2 = \left(\frac{1}{2}\alpha + \beta^2\right) mR^2\,.$$

The final moment of inertia after Slavo's arrival is

$$I_1 = I_0 + I_\mathrm{S} = \left(\frac{1}{2}\alpha + \beta^2 + 1\right) mR^2\,.$$

We will find the final angular frequency by the law of conservation of angular momentum

$$\begin{aligned}
L_0 &= L_1\,,\\
I_0\omega_0 &= I_1\omega_1\,,\\
\omega_1 &= \omega_0\frac{I_0}{I_1}\,,\\
\omega_1 &= \omega_0\frac{\frac{1}{2}\alpha + \beta^2}{\frac{1}{2}\alpha + \beta^2 + 1}\,,\\
\omega_1 &\doteq 0.848\omega_0 \doteq 1.02\,\mathrm{rad{\cdot}s^{-1}}\,.
\end{aligned}$$

After Slavo's arrival, the disc flywheel will have angular frequency $\omega_1 \doteq 1.02\,\mathrm{rad{\cdot}s^{-1}}$.

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Problem 4 ... roofing

3 points

Martin is standing on a roof. The coefficient of static friction between his shoes and roof tiles is 0.7. To what percentage of the initial value does the effective coefficient of friction between Martin and the roof decrease when Martin sits down, if doing so transfers 60 % of his weight from shoes to his trousers? The coefficient of static friction between his trousers and roof tiles is 0.4.

Martin was learning to be a roofer.

In the first case, the effective coefficient is clearly $f_1 = 0.7$. In the second case, $1 - w = 40\,\%$ of Martin's weight takes the coefficient $f_1 = 0.7$ and $w = 60\,\%$ of his weight takes the coefficient $f_2 = 0.4$. Thus, the effective coefficient in the second case is

$$f = 0.4 \cdot 0.7 + 0.6 \cdot 0.4 = 0.52\,.$$

We assume constant roof inclination at the spot where Martin stood and sat. We could express the resulting coefficient as

$$f = \frac{F_\mathrm{f}}{F_\mathrm{G}} \frac{f_1 F_1 + f_2 F_2}{F_1 + F_2} = f_1(1-w) + f_2 w\,.$$

The asnwer is the ratio of the two values $\frac{0.52}{0.7} \doteq 0.74$, what is approximately $74\,\%$.

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Problem 5 ... spring in an airplane 3 points

An ideal spring oscillator is placed in a climbing airplane that has a load factor 3g. How does the frequency of its oscillations change? Express your answer as the ratio of the original frequency to the new one. *Vojta was oscillating in an airplane.*

The frequency of an ideal spring oscillator depends only on the spring stiffness and mass of a suspended object; therefore, the frequency does not change when put airborne. Thus, the answer is 1.00 times.

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Problem 6 ... focusing light 4 points

We have an aquarium and a thin converging lens with a small diameter, which has an optical power of $\varphi = 4.20\,\mathrm{D}$ in the air. We place a point light source on the optical axis of the lens at a great distance from the lens. The lens is placed $d = 4.20\,\mathrm{cm}$ from the aquarium in such a way that its optical axis is perpendicular to the wall of the aquarium. The aquarium is large enough, its walls are made of thin glass, and it is full of water. What is the distance from the center of the lens at which the rays coming from the source are focused?

Karel was thinking about optics.

We can assume, that the light falling on the lens comes from infinity, the refracted rays therefore point to the focus. So if there were no aquarium, answer would be, that the rays intersect in the distance

$$f = \frac{1}{\varphi} \doteq 23.8\,\mathrm{cm}\,.$$

However in our situation, rays refract again in the distance d. Water is optically denser medium, refracted rays will thus be closer to normal, which means that the point of their intersection will be located further than in air. Thanks to the small diameter of the lens, we are only interested in rays close to optical axis and so we can guess[1], that the ratio of those distances will match the ratio of indices of refraction. So we get

$$\frac{x}{f-d} = \frac{n}{n_0} \quad \Rightarrow \quad x = \frac{n}{n_0}\left(f-d\right)\,,$$

[1] This is faster and less correct variant of the solution. Better explanation follows.

where x denotes the distance from the aquarium wall in which the rays intersect. The total distance from the lens to the point of intersection is thus

$$s = d + x = d + \frac{n}{n_0}(f - d) \doteq 30.3\,\mathrm{cm}\,.$$

Rays intersect 30.3 cm from the optical center of the lens. With the required precision, we considered air's index of refraction to be equal to the index of refraction of vacuum, that is $n_0 \doteq 1$. Let's now look at more detailed explanation of why does our solution work.
We will analyze the whole situation with a help of a picture 1, in which the angles are highlighted for better idea. Axis o is the optical axis of our lens.

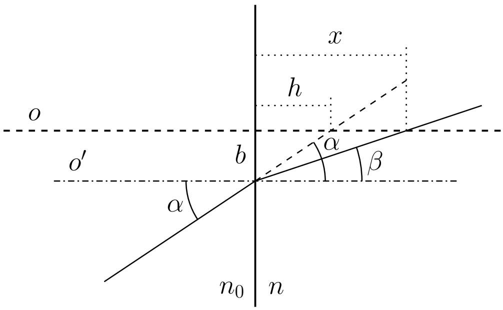

Fig. 1: Schematic representation of the refraction at the interface of air (n_0) and aquarium (n) – dashed line o denotes axis of the lens, o' denotes the normal of incidence, full line illustrates the ray, $h = f - d$ is the distance, at which would the rays at the same distance from axis o (denoted as b) intersect without an interface, and x is the distance of the actual intersection

We're interested in what happens at the interface of air and water. Although the aquarium is made out of glass and thus there should be two refractions, its wall is thin so the first refraction can be neglected. The ray arrives at the optical interface at the angle α in the distance b from the lens.

The normal is denoted as o' and the angle of refraction is β. If there were no aquarium, the rays would have intersected at the distance $f - d$ from the interface. In our situation, they will however intersect in the distance again denoted as x.

So we've described the picture and now let's focus on the solution itself. Consider two specific triangles in the picture, from which we can express the tangents of angles α and β as follows

$$\tan\alpha = \frac{b}{f-d}\,, \quad \tan\beta = \frac{b}{x}\,.$$

We rearrange the second expression to $b = x\tan\beta$, which we substitute into the first one and then solve for x

$$x = (f - d)\,\frac{\tan\alpha}{\tan\beta}\,.$$

Now we prepare the Snell's law (law of refraction) into the form suitable for substitution.

$$n_0 \sin\alpha = n \sin\beta \qquad \Rightarrow \qquad \beta = \arcsin\left(\frac{n_0}{n}\sin\alpha\right)\,.$$

$$x = (f-d)\,\frac{\tan\alpha}{\tan\left(\arcsin\left(\frac{n_0}{n}\sin\alpha\right)\right)}\,.$$

At this point, it would be useful to use following identity, which holds for all z from the domain of $\arcsin x$ function,

$$\tan\arcsin z = \frac{z}{\sqrt{1-z^2}}\,.$$

In our situation $z = \frac{n_0}{n}\sin\alpha$. We'll also make use of $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and we get

$$\begin{aligned} x &= (f-d)\tan\alpha\left(\frac{\frac{n_0}{n}\sin\alpha}{\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha}}\right)^{-1} \\ &= (f-d)\,\frac{n}{n_0}\,\frac{\frac{\sin\alpha}{\cos\alpha}}{\sin\alpha}\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha} \\ &= \frac{n}{n_0}\,(f-d)\,\frac{1}{\cos\alpha}\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha}\,. \end{aligned}$$

By this, we've even obtained an exact result for rays falling on the interface at an arbitrary angle. However now we can notice, that thanks to the small diameter of the lens, all the rays are close to the axis o. Angle α is thus small, so we could set $\alpha \approx 0$, which also means $\sin\alpha \approx 0$ and $\cos\alpha \approx 1$. So we get

$$x = \frac{n}{n_0}\,(f-d)\ ,$$

which is exactly what we wanted to show. The rest of the solution, that is adding the distance from the lens to the aquarium, is then analogous.

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Problem 7 ... induced voltage 3 points

Consider a region with homogeneous magnetic field. The region has a rectangular cross-section with sides $a = 3.0\,\mathrm{m}$, $b = 2.0\,\mathrm{m}$ and the magnetic induction vector $B = 1.0\cdot 10^{-3}\,\mathrm{T}$ is perpendicular to this cross-section. We place a straight wire in parallel to the side a, such that its free ends lie outside the region with the magnetic field. We connect these ends to a voltmeter.

We then move the wire in uniform linear motion at velocity $v = 0.20\,\mathrm{m{\cdot}s^{-1}}$ along the side b, i.e. in a direction perpendicular to the magnetic induction vector and to the side a. What voltage does the voltmeter show when the wire passes through the magnetic field?

Jindra came up with a problem longer than its solution.

The voltage is calculated as the ratio of a change in the magnetic induction flux $\mathrm{d}\Phi$ through the loop with respect to an infinitesimal time change $\mathrm{d}t$

$$U = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}SB}{\mathrm{d}t} = \frac{\mathrm{d}x\cdot aB}{\mathrm{d}t} = Bva = 6.0\cdot 10^{-4}\,\mathrm{V}\,.$$

The same result can be obtained by using the formula for induced voltage $U = Bvl$.

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Problem 8 ... a constant atmosphere

3 points

What would be the thickness of the Earth's atmosphere if it had constant density $\rho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$ everywhere? Assume that the Earth is round and that the atmosphere has a mass of $m = 5.157 \cdot 10^{18}\,\mathrm{kg}$.

Karel is always speculating.

The volume of the atmosphere of a constant density ρ is

$$V = \frac{m}{\rho} \doteq 3.998 \cdot 10^{18}\,\mathrm{kg}\,.$$

We assume the Earth's radius to be $R_\oplus = 6.378 \cdot 10^6\,\mathrm{m}$. While it is the equatorial radius, it is sufficient enough for the required accuracy of this computation. By using the radius, we get the Earth's surface area $S = 4\pi R_\oplus^2 \doteq 5.11 \cdot 10^{14}\,\mathrm{m}^2$.

Regarding the atmosphere being relatively thin, we can assume the top and the bottom surface area of the atmosphere to be the same. The thickness of the atmosphere h is then the ratio of the atmosphere's volume and Earth's surface area

$$h = \frac{V}{S} = \frac{m}{4\pi\rho R_\oplus^2}\,,$$

$$h \doteq 7.8 \cdot 10^3\,\mathrm{m}\,.$$

Thus, the constant atmosphere would be of thickness $7\,800\,\mathrm{m}$.

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Problem 9 ... molar mumble

4 points

An enthomologist caught a "mol" and he wants to dry it. Calculate its mass after evaporation of all the water, if you know that a particular ball of one mole of living "mols"

- *contains $n = 3.346 \cdot 10^{19}\,\mathrm{mol}$ of water molecules,*
- *has surface area equal to twenty four times the area of Moldova (including Transnistria),*
- *has density $\rho = 27.4\,\mathrm{kg{\cdot}m^{-3}}$.*

Translator's note: "mol" is a Czech word for a clothing moth.

Vojta was admiring Vivaldi's Summer's tremolos.

Let N be the number of "mols" in the ball (numerical value of Avogadro's constant in mol^{-1}). First we will compute the mass of a living *mol*

$$m_\mathrm{m} = \frac{\rho V}{N_\mathrm{A}} = \frac{\rho \frac{4}{3}\pi r^3}{N_\mathrm{A}}\,,$$

whereas

$$r = \sqrt{\frac{24 S_\mathrm{M}}{4\pi}} = \sqrt{\frac{6 S_\mathrm{M}}{\pi}}\,,$$

where $S_\mathrm{M} = 33\,843\,\mathrm{km}^2$ is the area of Moldova. Overall we have

$$m_\mathrm{m} = \frac{4\pi\rho}{3N_\mathrm{A}}\left(\frac{6S_\mathrm{M}}{\pi}\right)^{\frac{3}{2}} = \frac{8\rho}{N_\mathrm{A}}\sqrt{\frac{6S_\mathrm{M}^3}{\pi}} \doteq 3.13\,\mathrm{mg}\,.$$

The mass of the water in the *mol* will be

$$m_\mathrm{w} = \frac{nM_{\mathrm{H_2O}}}{N_\mathrm{A}} \doteq 1.00\,\mathrm{mg}\,,$$

Thus

$$m = m_\mathrm{m} - m_\mathrm{w} = 2.13\,\mathrm{mg}\,.$$

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Problem 10 ... don't lean out of the windows 4 points

A dignitary is taking a fiacre ride. While moving at a speed $v_\mathrm{k} = 2.0\,\mathrm{m{\cdot}s^{-1}}$, he's eating an apple. When finished eating, he throws the apple core out of the window such that it gains horizontal velocity $v_\mathrm{h} = 3.0\,\mathrm{m{\cdot}s^{-1}}$ perpendicular to the direction of travel. The core then proceeds to hit an unsuspecting peasant in the head just when he is $d = 1.0\,\mathrm{m}$ away from the dignitary and is not moving. At what speed did the apple core hit the peasant's head? Consider the core to be a point mass. *Lego often travels by train.*

The core's velocity will consist of three components, which can be considered independent. The first component will be the speed v_k in the direction of travel (let us denote this direction as x). The second component will represent the speed v_h at which it was thrown out of the window (in the horizontal direction perpendicular to x – let's denote this direction as y) and the third, vertical component (the direction z) gained thanks to the gravity.

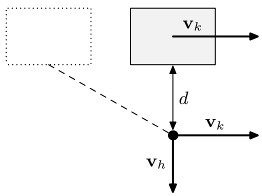

Fig. 2: Illustration of fiacre's and apple core's movements.

We know that the distance between peasant and fiacre at the time of impact was equal to d. Note that the core and the fiacre must have traveled the same distance in the x direction. This implies that in the time of impact, the distance between fiacre and peasant must have been minimal (see the illustration 2), which means the core must have traveled exactly distance d in the y direction. Thanks to this observation, we're able to compute the time of its flight $t = \frac{d}{v_\mathrm{k}}$, from which we can subsequently determine the speed in z direction

$$v_z = \frac{gd}{v_\mathrm{h}}\,.$$

Now we can finally use the Pythagorean theorem to compute the final speed as

$$v = \sqrt{\left(\frac{gd}{v_\mathrm{h}}\right)^2 + v_\mathrm{k}^2 + v_\mathrm{h}^2} \doteq 4.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 11 ... a healthy mind in a healthy body 4 points

Let a flexible rope be fastened to a wall at one end. Danka is pulling the other end. She is standing on a small rug that lies freely on the floor. What is the maximum distance (measured from the wall) at which Danka can be pulling the rope, such that the elasticity of the rope does not drag Danka back towards the wall? The coefficient of static friction between the rug and the floor is $f = 0.45$, while it is much larger between Danka's feet and the rug. Danka weighs $m = 55\,\mathrm{kg}$ and the mass of the rug can be considered negligible. Assume that the rope is being pulled horizontally, its free length is $l_0 = 1.97\,\mathrm{m}$ and its stiffness is $k = 164\,\mathrm{N{\cdot}m^{-1}}$.

Danka was exercising.

An elastic force of a rope is given by the extension with respect to the proper length as $F_\mathrm{e} = k(l - l_0)$, where l is the length of the rope. Static friction force grows with an increasing elastic force in a way that both forces cancel out. However, this applies only until the point when we reach a maximum static friction force, which we can calculate as

$$F_\mathrm{f} = fF_\mathrm{n} = fmg\,.$$

where F_f is a friction force in a direction opposite to the force exerted by the rope. F_n is the force exerted by Danka and the rug on the ground. As we know, there exists a critical point where the elastic force finds its balance with the maximum static friction force

$$k(l - l_0) = fmg\,.$$

We isolate the length of the rope and plug in the corresponding values

$$l = l_0 + \frac{fmg}{k}\,,$$

$$l \doteq 3.45\,\mathrm{m}\,.$$

Danka can extend up the rope to 3.45 m from the wall.

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Problem 12 ... insidiously smoky 4 points

A typical ionization smoke detector contains a sample of $^{241}\mathrm{Am}$ that corresponds to a radioactivity of $1\,\mathrm{\mu Ci}$. How many such detectors at the minimum do we need to dismantle to get a sufficient mass of $^{241}\mathrm{Am}$ to start a nuclear chain reaction? The critical mass of $^{241}\mathrm{Am}$ corresponds to $60\,\mathrm{kg}$.

Pepa always steals problems from textbooks.

We look up the half-life of $^{241}\mathrm{Am}$ in the tables, $T = 432.6$ years. Specific activity of the sample can be calculated as

$$a = \frac{1}{m}\left|\frac{\mathrm{d}N}{\mathrm{d}t}\right| = \frac{\lambda N}{m} = \frac{\lambda N}{NA_\mathrm{R}u}\,,$$

where $A_\mathrm{R} = 241$ is relative atomic mass of $^{241}\mathrm{Am}$, $u = 1.661\cdot 10^{-27}\,\mathrm{kg}$ is the atomic mass constant, and λ is the exponential decay constant of $^{241}\mathrm{Am}$, i.e. $\frac{\ln 2}{T}$.

We need to convert the activity of the sample to units of SI, which is Becquerel ($\mathrm{Bq} = \mathrm{s}^{-1}$). Becquerel is related to Curie (Ci) as follows: $\mathrm{Ci} = 3.7 \cdot 10^{10}\,\mathrm{Bq}$. The reason we had to convert the units was because we determined the specific activity of americium in units $\mathrm{Bq \cdot s^{-1}}$. Then, for a mass of one sample from the detector we get

$$m = \frac{3.7 \cdot 10^4\,\mathrm{Bq}}{a} \doteq 0.29\,\mathrm{\mu g}\,.$$

The critical mass of 60 kg thus corresponds to approximately $60\,\mathrm{kg}/0.29\,\mathrm{\mu g} = 207$ billion detectors.

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Problem 13 ... an electron source 4 points

Consider a point source that emits electrons of speed $v = 20\,000\,\mathrm{km \cdot s^{-1}}$ in the plane yz, and is placed in a homogeneous magnetic field which has magnitude B and is pointing in the direction of the x axis. The electron source is surrounded by a tube of radius $R = 10\,\mathrm{cm}$. The axis of symmetry of the tube passes through the electron source and is parallel to the y axis. What is the minimum magnitude of the magnetic field B which prevents the emitted electrons from touching the tube? *Kiko has been reading famous "green lecture notes".*

The magnetic component of the Lorentz force acting on an electron is

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B}\,,$$

where $e \doteq 1.6 \cdot 10^{-19}\,\mathrm{C}$ is the elementary charge. Magnetic induction $\mathbf{B}$ is oriented in the direction of x axis, i.e. perpendicular to the arbitrary velocity vector $\mathbf{v}$ of plane yz. It follows that for the magnitude of Lorentz force we have

$$F = evB\,.$$

The Lorentz force always lies in the plane yz and is perpendicular to the velocity $\mathbf{v}$. It means that electron will move in a circle of the radius r. For the centripetal acceleration, the following equation holds

$$\frac{v^2}{r} = a = \frac{F}{m_e} = \frac{eBv}{m_e}\,.$$

To prevent the electrons from touching the tube, the electron trajectory radius r can be at most half of the radius of the tube R. Now, we can express the minimum magnitude of the magnetic field

$$B = \frac{2m_e v}{eR} \doteq 2.3\,\mathrm{mT}\,.$$

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Problem 14 ... is it cold outside? 4 points

We are designing a thermometer. We want its range to be from $-40\,^{\circ}\mathrm{C}$ *up to* $+50\,^{\circ}\mathrm{C}$*, with the scale lines at every* $1\,^{\circ}\mathrm{C}$*. To distinguish the values on the thermometer properly just by eye from a distance of* $2\,\mathrm{m}$*, we need the angular distances between adjacent lines to be at least* $4'$*. The radius of the capillary tube of the thermometer is* $0.2\,\mathrm{mm}$*. What is the minimum volume of alcohol that the thermometer needs to contain if it has to be able to show the full range of temperatures? We use alcohol with the volumetric thermal expansion coefficient* $\beta = 1.1 \cdot 10^{-3}\,\mathrm{K}^{-1}$*.* *Jarda will sell you a thermometer one day.*

The spacing between the scale lines must be at least

$$d = l \tan\theta \approx l\theta \doteq 2.3\,\mathrm{mm}\,,$$

where $l = 2\,\mathrm{m}$ is the distance we are looking from and $\theta = 4'$ is the angle at which we have to see the gap between the lines. Let $\delta = 1\,^{\circ}\mathrm{C}$ be the smallest distance between two scale lines.

To be able to display the full range of temperatures, the thermometer has to measure at least

$$h = d\frac{\Delta T}{\delta} \doteq 21\,\mathrm{cm}\,.$$

The alcohol has to be able to fill out the volume of

$$\Delta V = \pi r^2 h \doteq 26\,\mathrm{mm}^3\,,$$

as the temperature changes by $\Delta T = 90\,^{\circ}\mathrm{C}$. The change of volume with respect to temperature is $\Delta V = V\beta\Delta T$, V being the initial volume and $\beta = 1.1 \cdot 10^{-3}\,\mathrm{K}^{-1}$ is the given volumetric thermal expansion coefficient of the alcohol. Thus

$$V = \frac{\Delta V}{\beta\Delta T} = \frac{\pi r^2 l\theta}{\beta}\,,$$
$$V \doteq 266\,\mathrm{mm}^3\,.$$

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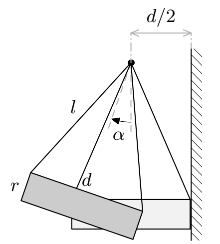
Problem 15 ... battering ram 5 points

We decided to conquer the Prague Castle. We will destroy the front gate with a battering ram in the shape of a thick homogeneous cylinder of length $d = 2.0\,\mathrm{m}$ *and radius* $r = 25\,\mathrm{cm}$*, with its axis placed horizontally. We attach cables with lengths* $l = 2.0\,\mathrm{m}$ *to the upper points of both bases of the cylinder and fasten the other ends to a common point right above the center of the cylinder. With the ram assembled in this way, we come to the gate and place the ram in such a way that one of its bases touches the gate. Then we incline the ram* $\alpha = 20^{\circ}$ *around the hanging point and finally release it. At what speed will it hit the gate?*



Jarda wants to be the president of FYKOS.

We can proceed from the law of conservation of energy. When we deflect the ram, we increase the position of the cylinder's center of gravity a bit, so we increase the potential energy

$$E_\mathrm{p} = mgh = mgs\left(1 - \cos\alpha\right) ,$$

where $\alpha = 20^\circ$ and $s = r + \sqrt{l^2 - \frac{d^2}{4}}$ is the distance of the center of gravity from the axis of its rotation. This energy is converted to rotational $E_\mathrm{r} = \frac{1}{2}J\omega^2$, where ω is the angular frequency of a rotation and J is the moment of inertia of the cylinder relative to the selected axis, which we yet need to calculate. The cylinder can still be approximated as a thin rod at given dimensions, so its moment of inertia with respect to the axis passing through the center of gravity perpendicular to the axis of symmetry is

$$J_s = \frac{1}{12}md^2 .$$

The error of this approximation is about five percent, but because we still have to use Steiner's parallel axis theorem, this error will no longer be important. The resulting moment of inertia by Steiner's theorem is

$$J = J_s + ms^2 = \frac{1}{12}md^2 + m\left(r + \sqrt{l^2 - \frac{d^2}{4}}\right)^2 .$$

The speed at which the battering ram hits the gate is then

$$v = \omega s = \sqrt{\frac{2mgs\left(1 - \cos\alpha\right)}{J}}s = \sqrt{\frac{2gs\left(1 - \cos\alpha\right)}{\frac{1}{12}d^2 + \left(r + \sqrt{l^2 - \frac{d^2}{4}}\right)^2}}s = 1.47\,\mathrm{m\cdot s^{-1}} .$$

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Problem 16 ... night-time thermodynamics lesson 4 points

Verča got lost on a trip and decided to stay in an old shack overnight. The outside temperature dropped to $t_0 = 10.0\,^\circ\mathrm{C}$ at night, but Verča kept the inside temperature at $t_1 = 16.0\,^\circ\mathrm{C}$. However, the shack had a rectangular hole in the wall of dimensions $a \times b$, where $a = 0.50\,\mathrm{m}$ and $b = 0.30\,\mathrm{m}$. To save some of the escaping heat, Verča plugged the hole with two bricks of cross-sections $S_1 = a \times (b/3)$ and $S_2 = a \times (2b/3)$, and of thermal conductivity coefficients $\lambda_1 = 0.80\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$ and $\lambda_2 = 1.30\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$, respectively. The lengths of both bricks are the same as the thickness of the wall $s_1 = 15\,\mathrm{cm}$, so they filled in the hole perfectly. Nevertheless, it would be nicer to be able to look outside. What thermal conductivity coefficient λ would a homogeneous glass panel of thickness $s_2 = 3.0\,\mathrm{mm}$ need to have if it conducted heat from the shack in the same way as the two bricks? The glass panel would be inset in the hole parallelly to the wall (like a window). Assume that the rest of the shack insulates perfectly.

Verča gets lost in other places than just lectures.

The quantity we want to preserve is the thermal flux, which we can generally calculate as

$$q = \frac{t_1 - t_0}{R} \,,$$

where $t_1 - t_0$ is the temperature difference and R is the thermal resistance of the body

$$R = \frac{d}{\lambda S} \,,$$

while d is the thickness of the body through which we conduct the heat, S is the cross-section of the body, and λ is its coefficient of thermal conductivity. Because the temperature difference is the same in both cases, we only need to compare the thermal resistances of the bricks and the glass block. The resistances compound in the same way as in electrical circuits, so for the total thermal resistance of the parallelly laid bricks, the following expression holds

$$\frac{1}{R_{\mathrm{c}1}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{ab(\lambda_1 + 2\lambda_2)}{3s_1} \quad \Rightarrow \quad R_{\mathrm{c}1} = \frac{3s_1}{ab(\lambda_1 + 2\lambda_2)}$$

and the thermal resistance of the glass block is

$$R_{\mathrm{c}2} = \frac{s_2}{ab\lambda} \,.$$

From the equality

$$R_{\mathrm{c}1} = R_{\mathrm{c}2}$$

we get

$$\frac{3s_1}{2\lambda_2 + \lambda_1} = \frac{s_2}{\lambda} \,,$$

from which we can express the result

$$\lambda = \frac{s_2(2\lambda_2 + \lambda_1)}{3s_1} \,.$$

We can see that nothing but the values of the individual factors and thicknesses affected the result, as all other quantities were the same in both cases. Numerically we get $\lambda = 0.022\,7\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$.

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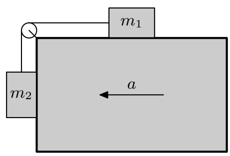
Problem 17 ... moving pulleys on a wagon

5 points



We have a wagon in the shape of a cuboid, which is moving with an acceleration of $a = 1.2\,\mathrm{m{\cdot}s^{-2}}$. On the top of the wagon, a smaller cuboid of mass $m_1 = 15\,\mathrm{kg}$ is placed, and it is connected by a rope over a pulley with another cuboid of mass $m_2 = 10\,\mathrm{kg}$, which is hanging on the rope in front of the wagon, touching its front wall. With what acceleration (with respect to the wagon) are the small cuboids going to move? The static friction coefficient between the small cuboids and the wagon is $f = 0.15$. Assume that the rope and pulley are massless.

Legolas truly enjoys adjusting his own problems.

Since the rope and the pulley are massless, the magnitude of the force by which the rope pulls the block m_2 upwards is the same as the magnitude of the force that pulls the cuboid m_1 to the left. Let's denote the magnitude of this force as T.

In addition to the force T, friction acts on the cuboid m_1 (the normal force from the wagon and the gravitational force cancel each other out), too. The magnitude of the force of friction is $F_\mathrm{t}^1 = fm_1g$. This force acts against acceleration, so the resulting equation of motion for this cuboid will be

$$T - fm_1g = m_1a_1 \,,$$

where we assume that the block will accelerate more than the wagon and thus will slide to the left. If our assumption were incorrect, we would end up with a negative result. But beware! It would not be enough to simply change its sign, as it would mean that we assumed the wrong direction of acceleration all the time, and we would expect the opposite sign of friction in the entire subsequent calculation! Why do we assume this direction? From a simple estimate that $m_1a < m_2g$.

In the system accelerating together with the wagon, the cuboid m_2 presses against the wall of the wagon with force m_2a, which is also the force by which the wall pushes it back (it means that in the system connected to the wagon, this block will not accelerate horizontally, as we would expect). Thus, the friction force $F_\mathrm{t}^2 = fm_2a$, the gravitational force $F_\mathrm{g}^2 = m_2g$, and the tensile force of the rope T act on the cuboid in the vertical direction. Consistently with the first cuboid, let's assume that this one also accelerates downwards, and we obtain the equation

$$m_2g - T - fm_2a = m_2a_2 \,.$$

In these two equations, the accelerations and the force T are unknown, but since both blocks are tied to one tensioned rope, their accelerations will be the same in the system connected to the wagon. However, a_1 is the resulting acceleration, so we still need to express the acceleration of the cuboid m_1 in the system connected to the wagon a_1'. We can do this either by adding a fictitious force or by noticing that $a_1 = a + a_1'$. Either way, we will get

$$T - fm_1g - m_1a = m_1a_1' \,.$$

Subsequently, we use the information that in this system $a_2 = a_1'$, we add up the equations to eliminate the unknown T and express the resulting acceleration as

$$m_2g - m_1a - fm_2a - fm_1g = (m_1 + m_2)\,a_2$$

$$a_2 = \frac{m_2g - m_1a - fm_2a - fm_1g}{m_1 + m_2} \doteq 2.2\,\mathrm{m{\cdot}s^{-2}} \,.$$

Of course, we could have started working in an accelerating system with the wagon since the beginning, in which $m_{1,2}a$ would be interpreted as the inertial force acting on the bodies in this system. After a very similar (and probably a little shorter) calculation, we would reach the same result.

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Problem 18 ... beware of the fountain

4 points

There is a circular fountain at a town square with a diameter of $D = 6\,\mathrm{m}$. At its center, a spring of water gushes vertically upwards to a height of $h = 4\,\mathrm{m}$. What is the lowest speed a short gust of wind has to blow at so that the gushing water falls outside the fountain? Assume that the wind blows horizontally and transfers $20\,\%$ of its speed to water drops.

Danka was observing children swimming in a fountain.

The longer the drops of water will be in the air, the further they can fly. Hence, it is best for the gust of wind to blow right after the water gushes out. This way, the drops will gain a horizontal component of velocity $v = 0,2\,v_w$, where v_w is the speed of wind we are searching. The time T that the waterdrop spends above ground is double the time t it takes for it to fall from the highest point h of its trajectory so that we can express it as

$$T = 2t = 2\sqrt{\frac{2h}{g}}\,,$$

where g is the gravitational acceleration. The borderline case, where the drop hits directly the edge of the fountain occurs, if the drop surpasses in the horizontal direction the radius of the fountain, that is to say $vT = D/2$. Therefore, the wind must blow at minimum at a speed of

$$v_w = \frac{5}{4}\sqrt{\frac{g}{2h}}D \doteq 8.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 19 ... Beryllgläser

5 points

We have two thin lenses of the same shape. The first one is made out of plastic with a refractive index of $n_\mathrm{p} = 1.67$ and has an optical power of $\varphi_\mathrm{p} = 4.20\,\mathrm{D}$. The second lens is made out of beryl and has a refractive index of $n_\mathrm{b} = 1.57$. What is the optical power of the second lens? Both of the lenses are surrounded by air.

Karel heard that eyeglasses were made out of beryl in old times, which inspired its German name, Brille.

We can compute the optical power for a thin lens as

$$\varphi = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,,$$

where n_2 is the refractive index of the surrounding environment, in our case air with a refractive index of $n_\mathrm{a} = 1$, n_1 is the refractive index of a lens (either plastic or beryl) and r_1, r_2 are the radii of curvature of the lens. However, radii aren't really important for us, because we know, that both of the lenses have the same shape; therefore, the value $(1/r_1 + 1/r_2)$ remains the same for both lenses. We can determinate this expression from the value of the optical power of the plastic lens

$$\varphi_\mathrm{p} = (n_\mathrm{p} - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$
$$\frac{\varphi_\mathrm{p}}{n_\mathrm{p} - 1} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,.$$

Now we only need to subtitute this into the expression for the beryl lens

$$\varphi_\mathrm{b} = (n_\mathrm{b} - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = (n_\mathrm{b} - 1)\frac{\varphi_\mathrm{p}}{n_\mathrm{p} - 1} = 3.57\,\mathrm{D}\,.$$

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Problem 20 ... filling the bathtub 3 points

Robert of mass $m = 60\,\mathrm{kg}$ and of the same density as water wanted to measure the volumetric flow rate of water in the tap while filling his bathtub. The bathtub has a rectangular base of dimensions $150\,\mathrm{cm} \times 75\,\mathrm{cm}$. Robert filled the bathtub with water to the height $h = 20\,\mathrm{cm}$, when he had $p = 80\,\%$ of his body under the water. It took him $t = 8\,\mathrm{min}$ to fill the bath. Find the volumetric flow rate of water in the tap. *Robert spends too much time in the bathtub.*

The volume of the bathtub to the water level is $V_\mathrm{b} = abh = 0.225\,\mathrm{m}^3$. However, Robert dislodges the water of the same volume as his submerged part of his body, which is $V_\mathrm{s} = pm/\rho = 0.048\,\mathrm{m}^3$. The volume of the water in the bathtub is $V = V_\mathrm{b} - V_\mathrm{s} = 0.177\,\mathrm{m}^3$.

Thus, the volumetric flow rate of the water in the tap is

$$Q = \frac{V}{t} \doteq 22\,\mathrm{l{\cdot}min^{-1}}\,.$$

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Problem 21 ... Lipno and relativity 5 points

Jarda found out that running is physically demanding. He wants to accelerate himself to the speed of $99\,\%$ of the speed of light in vacuum; however, he would need lots of energy. To imagine such an amount of energy, he calculated how many times he would have to heat and evaporate the entire Lipno reservoir, which has a volume of 310 million cubic meters. Assume the initial water temperature is always $20\,^\circ\mathrm{C}$. Jarda weighs $75\,\mathrm{kg}$. If Jarda's calculations are correct, what number did he get? *Jarda ran out of energy.*

At the speed v which is so close to the speed of light c, the total energy of a moving object is given by a relativistic relation

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}\,,$$

where m_0 is the rest mass. Now that Jarda is at rest and he's writing a solution to Physics Brawl Online problem, his energy is $E_0 = m_0 c^2$, so the energy he needs to accelerate himself is "only"

$$\Delta E = E - E_0 = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) = 6.09 m_0 c^2\,.$$

Now, we are going to calculate how much heat is needed for one cycle of heating and evaporation of Lipno. It's the sum

$$Mc_\mathrm{w}\Delta T + Ml = \rho V \left(c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + l\right) ,$$

where $M = \rho V$ is the mass of water in Lipno, $V = 310 \cdot 10^6\,\mathrm{m^3}$ is the volume of Lipno, $\rho = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water, c_w is specific heat capacity of water, l is the enthalpy of vaporization of water and $T_\mathrm{w} - T_0 = 100\,°\mathrm{C} - 20\,°\mathrm{C} = 80\,°\mathrm{C}$ is the required temperature difference. In the end, we just express the number of these evaporation cycles as

$$n = \frac{6.09 m_0 c^2}{\rho V \left(c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + l\right)} = 51.3 \doteq 52 \,.$$

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Problem 22 ... the intersections of Lissajous curves — 5 points

There is a point that oscillates along the x axis according to the equation $x = x_\mathrm{m}\sin 5\omega t$ and along the y axis according to $y = y_\mathrm{m}\sin 6\omega t$, where x_m and y_m are amplitudes (which are not necessarily the same), t is time and 5ω and 6ω are angular frequencies of oscillations. How many intersection points of the motion graph are there in the xy plane?

Karel loves to create problems on Lissajous figures.

The quickest approach to finding the solution is to plot a graph in arbitrary software, supporting parametric equations. One of the most direct options is to use Wolfram Alpha [2], which is available for free. We plot a graph for $x_\mathrm{m} = y_\mathrm{m} = 1$ as these values only scale the height and width of our graph but don't change the number of intersections. We similarly consider $\omega = 1$, because it only tells us how fast the point travels through the graph. Due to the remark that the movement repeats to infinity, even very slow oscillations plot the whole graph. If we get a graph similar to the one we see in the figure 3, it only remains to calculate the points where the Lissajous curve intersects with itself. If we go by "intersection columns", we add $4+5+4+5+4+5+4+5+4+5+4 = 49$. If we took it line by line, we would get the same result $5+6+5+6+5+6+5+6+5 = 49$. So the graph of motion has altogether 49 points of intersection in different locations.

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Problem 23 ... conductor in a hurry — 5 points

At the beginning of every rehearsal session of the symphonic orchestra, all the players tune their instruments. The process begins with a piano or an oboe giving off the tone A (in this problem, we assume it has a frequency $f_\mathrm{A} = 443\,\mathrm{Hz}$), and the other musicians tune their instruments according to them. Imagine that the conductor is still running around during the process with velocity $v_\mathrm{c} = 3\,\mathrm{m{\cdot}s^{-1}}$. Suppose that when running from the oboist to the pianist, the sound waves of their instruments interfere. What beat frequency is he going to hear?

Vojta was tuning his cello unsuccessfully.

[2] https://tinyurl.com/lissajous-intersections

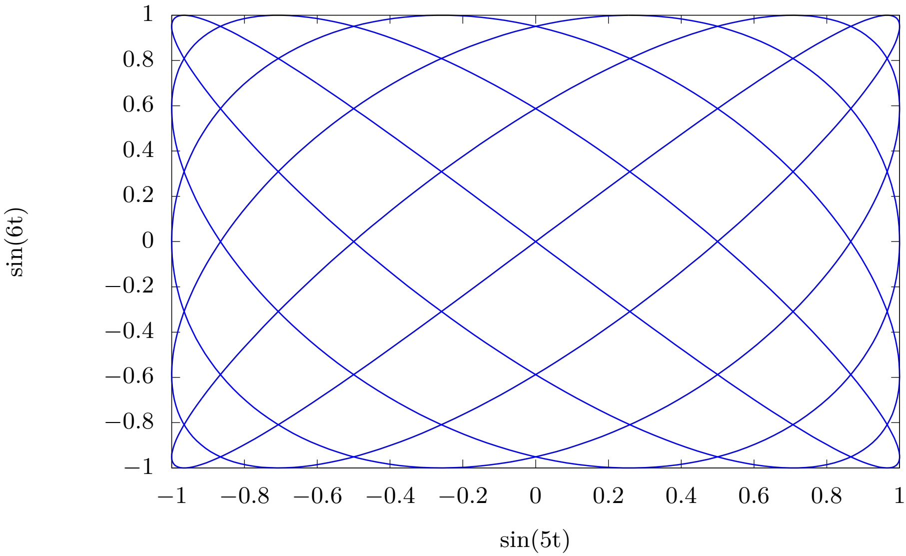

Fig. 3: Plotted Lissajous figure.

We will determine the beat frequency f_b as the difference of the two frequencies that reach the conductor

$$f_\mathrm{b} = f_\mathrm{p} - f_\mathrm{o} = f_\mathrm{A}\left(\frac{c + v_\mathrm{c}}{c}\right) - f_\mathrm{A}\left(\frac{c - v_\mathrm{c}}{c}\right) = 2f_\mathrm{A}\frac{v_\mathrm{c}}{c} \doteq 7.74\,\mathrm{Hz}\,.$$

Alternatively, we might interpret the situation as if the conductor was running along a standing wave formed by the interference of the two tones. The conductor then hears the beat every time he passes through one of the antinodes, which are half a wavelength apart, that is $\frac{c}{2f_\mathrm{A}}$. The beat frequency then is the inverse of time T required for the conductor to travel this distance

$$f_\mathrm{b} = \frac{1}{T} = 2f_\mathrm{A}\frac{v_\mathrm{c}}{c}\,,$$

which gives the same result.

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Problem 24 ... height does not matter 4 points

Consider a classical mathematical pendulum – a point mass on a massless string. Such a pendulum is placed in a hot air baloon and is left to rise through the atmosphere. The temperature t of the surrounding air decreases linearly with the height above the sea level h as $t = t_0 - kh$, where $t_0 = 25\,^{\circ}\mathrm{C}$ and $k = 0.007\,\mathrm{K{\cdot}m^{-1}}$. At the same time, however, the magnitude of the Earth's gravitational field slowly decreases as the height increases. We want the period of oscillations of the pendulum to remain independent of the height h. What should the (non-zero) value of the thermal expansion coefficient α of the string be?

Jarda gets cold on the 16th floor of dormitories.

A period of mathematical pendulum is

$$P = 2\pi\sqrt{\frac{l}{g}}\,,$$

where g is the acceleration due to gravity and l is the length of the string. However, the acceleration due to gravity decreases with the increasing altitude h, because the gravitational force decreases with increasing distance from the Earth's surface. Neglecting the centrifugal force leads to

$$g = \frac{GM}{r^2} = \frac{g_0 R_\oplus^2}{(R_\oplus + h)^2} \approx g_0\left(1 - \frac{2h}{R_\oplus}\right)\,,$$

where $R_\oplus$ is the Earth's radius and g_0 is the acceleration due to gravity at the Earth's surface.

If the period is to remain constant, while acceleration decreases due to gravity, the length of the string has to shorten as well. However, this already happens due to thermal expansion. The length of the pendulum changes linearly with temperature according to relation

$$l = l_0\left(1 - \alpha(t_0 - t)\right) = l_0\left(1 - \alpha(t_0 - t_0 + kh)\right) = l_0\left(1 - \alpha kh\right)\,,$$

where α is the thermal expansion coefficient we are searching for. Since the period does not change, it holds

$$\frac{l_0}{g_0} = \frac{l}{g} = \frac{l_0(1 - \alpha kh)}{g_0\left(1 - \frac{2h}{R_\oplus}\right)}\,,$$

which leads to

$$\alpha = \frac{2}{R_\oplus k} \doteq 4.5 \cdot 10^{-5}\,\mathrm{K^{-1}}\,.$$

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Problem 25 ... the hollow Earth has many variants 5 points

How much would the value of the gravitational acceleration at the Earth's surface decrease if our planet was only a spherical shell of the same outer radius the Earth has now and of thickness $D = 100.0\,\mathrm{km}$? Consider the spherical shell to be homogeneous and of the same density as the Earth's average density. The result should be the ratio of this acceleration to the actual gravitational acceleration acting at the Earth's surface.

Karel keeps returning to the topic of spheres.

The mass of the planet is spherically symmetrically distributed. Thus, at its surface, we can assume that the problem is equivalent to the situation where the mass is concentrated in its center of gravitation, from which we are distant $R_\oplus$ (radius of the Earth). With the use of Newton's formula, we obtain force acting on a body of mass m, or the acceleration, respectively

$$F = G\frac{mM_\oplus}{R_\oplus^2}\,, \quad a_\mathrm{g} = G\frac{M_\oplus}{R_\oplus^2}\,.$$

Because the planet's mass is the only subject to change (in a way that both cases share the same density, but the new "planet" will be hollow), then the original and new acceleration should differ only in volume. The volume of a hollow sphere expressed by D and $R_\oplus$ is

$$V_\mathrm{H} = \frac{4}{3}\pi\left(R_\oplus^3 - (R_\oplus - D)^3\right) = \frac{4}{3}\pi\left(3R_\oplus^2 D - 3R_\oplus D^2 + D^3\right)$$

The gravitational acceleration on the surface of the hollow planet would be

$$a_\mathrm{H} = \frac{V_\mathrm{H}}{V_\oplus}a_\mathrm{g} = \frac{3R_\oplus^2 D - 3R_\oplus D^2 + D^3}{R_\oplus^3}a_\mathrm{g} \doteq 0.046\,3\,g\,.$$

The gravitational acceleration on the surface of a hollow planet would be less than five percent of the actual gravitational acceleration at the Earth's surface. The thinner the shell, the more negligible the second and third terms in the sum become. If the second and the third terms are neglected, the result will differ from the second valid digit. Neglecting the third term alone will be reflected only in the fifth valid digit.

According to the assignment, we considered only gravitational acceleration (not the net acceleration usually denoted as gravity), as we neglected the centrifugal force from the Earth's rotation. If we wanted to think about the acceleration we would feel on such a planet, we had to know the specific position and speed of rotation. However, we were not provided any information on whether the imaginary planet would rotate in the same or different direction compared to the Earth. One could even deduce that if they wanted the shell-shaped planet to be stable, it should rotate at a lower rate thanks to its less weight.

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Problem 26 ... an oscillating pulley

5 points

Consider a weightless pulley hanging from the ceiling on a spring with stiffness $k = 80\,\mathrm{N/m}$. A weightless rope is passed over the pulley. While one end of the rope is attached to the ground, there is a body of mass $m = 1\,\mathrm{kg}$ attached to the other end. If we pull the body slightly downwards and then release it, what will the period of small oscillations of the system be?

Lego loves oscillations and pulleys, so he finally put them together.

If we denote the distance, along which we have to pull the body down to the ground, as x, the pulley lowers its position just by $x/2$. This comes from the idea that if we pulled both ends of the rope simultaneously by x, then the pulley would also change its position by x. However, we pulled only one end, while the other is fixed; therefore, the pulley lowers by hlaf the distance.

Thus, the spring prolongs by $x/2$ and the force, by which the spring pulls the pulley up, increases by $kx/2$. The question stands, how does the force by which the rope acts on the body increases. The pulley is weightless therefore the force acting on it is always zero. The same stands for the rope, thus the tension has to be the same at each point of the rope. It implies that both parts of the rope (hanging from the pulley) pull the pulley down by the same force. As we already know, the total force (acting on the pulley) increased by $kx/2$; thus the tension in the rope has to increase by half of this value, i.e. $kx/4$.

Finally, if we pull the body down by distance x, the acting force, by which the rope pulls it up, increases by $kx/4$. Thus the stiffness that the body feels is $(kx/4)/x = k/4$. We can substitute the result to the formula for the period of the linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k/4}} = 4\pi\sqrt{\frac{m}{k}} \approx 1.4\,\mathrm{s}\,.$$

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Problem 27 ... one increased, the other increased too 5 points

Lego was captured by cannibals. They imprisoned him in a hut, where he had an ideal DC voltage source, many perfect conductors and several adjustable resistors (rheostats). To stay alive, Lego must create a specific electric circuit. In such a circuit, there has to be at least one rheostat such that increasing its resistance causes the current flowing through some other rheostat (in the same circuit) to increase. What is the minimum number of rheostats this circuit needs to contain? If such a circuit does not exist, the answer should be 0.

Lego has no clue whether anyone could fill themselves up by him.

We will start by assuming the smallest possible number of rheostats (theoretically). For one rheostat, we can quickly realize that it won't be possible to meet the given conditions because if we increase the resistance on it, the current flowing through it will decrease (and there is no other rheostat in the circuit that would meet the second condition).

We can look at the case of 2 rheostats. If we connect them in series, we reduce the current flowing through both rheostats by increasing the resistance on either of them, so again, we do not meet the second condition. If we connect two rheostats in parallel, and we increase the resistance on one of them, the current flowing through it decreases, and the current flowing through the other rheostat remains unchanged. Even though we still don't have the right solution, you may intuitively feel that we are getting closer.

Let's move on to the case of 3 rheostats. We realize that exclusively serial or exclusively parallel connections will behave analogously to the case of two rheostats. If we connect two rheostats in series and the third one in parallel to them, we again get a situation identical to two rheostats connected in parallel. So we will evaluate another possible connection - two rheostats (let's note them R_1 and R_2), connected in parallel, and the third (R_3) will be connected in series. As the resistance of R_3 increases, the current flowing through the remaining two rheostats decreases, and the other way around, as the resistances of R_1 and R_2 decrease, the current through R_3 decreases (we do not consider the case when one of the pair R_1 and R_2 is zero, as the current would not change again, which is not the case we are looking for). However, let's think about what will happen to the current flowing through R_2, if we increase the resistance of R_1.

First, we need to express the current flowing through R_2 (let's denote it as I_2, and the other currents by analogy). The total current flowing through the circuit is then also equal to I_3. When the circuit splits into branches, the total current I_3 must divide between the two branches, and in the inverse ratio of resistances (a larger current will flow through the smaller resistance). The current $I_2 = I_3R_1/(R_1+R_2)$ will go through R_2. We still need to calculate the total current I_3. We can determine it as the ratio of the voltage U on the source and the total resistance. The total resistance is the sum of the resistance R_3 and the parallel connection, which has the resistance $R_1R_2/(R_1 + R_2)$. Thus, the total current is

$$I_3 = \frac{U}{R_3 + R_1R_2/(R_1 + R_2)} = U\frac{R_1 + R_2}{R_1R_3 + R_2R_3 + R_1R_2}\,.$$

From here, we can express the current flowing through R_2 as

$$I_2 = I_3\frac{R_1}{R_1 + R_2} = U\frac{R_1}{R_1 + R_2}\frac{R_1 + R_2}{R_1R_3 + R_2R_3 + R_1R_2} = U\frac{R_1}{R_1R_3 + R_2R_3 + R_1R_2}\,.$$

We would get the same result if we firstly calculated how the voltage distributes and then determined the ratio of the voltage on the parallel connection and R_2.

But back to the main question – is there any possibility that by increasing R_1, I_2 will increase, too? It seems we have no other choice but to differentiate the result with respect to R_1

$$I_2 = I_3\frac{R_1}{R_1 + R_2} = U\frac{R_1}{R_1 + R_2}\frac{R_1 + R_2}{R_1R_3 + R_2R_3 + R_1R_2} = U\frac{R_1}{R_1R_3 + R_2R_3 + R_1R_2}\,.$$

We obtain that the derivative I_2 with respect to R_1 is (for positive resistances) always positive, i.e., the current flowing through the resistance R_2 increases with the resistance R_1. So we have found the circuit that the cannibals were demanding from Lego, and we needed just three rheostats to build this circuit. At the same time, we verified (in the beginning of the solution) that it would not be possible to meet the required conditions for a lower number of rheostats. Hence, the correct answer to the problem is that the circuit must contain at least three rheostats.

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Problem 28 ... Doppler on his way home 6 points

Christian Doppler was on his way home. After a while, he noticed that people in both directions (the same as his and the opposite) walk at a speed of $v = 1.0\,\mathrm{m{\cdot}s^{-1}}$ and have distances of $l = 4.0\,\mathrm{m}$ between them. Doppler decided to take the chance to find the speed at which he should walk in order to meet as few people as possible on his way home. What is the minimum number of people that Doppler will meet on his way home if he's $d = 5.2\,\mathrm{km}$ away from his home? You may assume that $d \gg l$, do not consider relativistic effects. *Legolas is kind of autistic.*

Let v_D be Doppler's velocity. The time that passes between meeting two people walking in the opposite direction is

$$t_\mathrm{p} = \frac{l}{v + v_\mathrm{D}}\,.$$

Therefore, the frequency of meeting people walking in the opposite direction is

$$f_\mathrm{p} = \frac{1}{t_\mathrm{p}} = \frac{v + v_\mathrm{D}}{l}\,.$$

Similarly, we can express the frequency of meeting people walking in the same direction as

$$f_\mathrm{r} = \frac{1}{t_\mathrm{r}} = \frac{|v - v_\mathrm{D}|}{l}\,.$$

The number of people that Doppler has met is given by the product of the sum of these meeting frequencies and the total time of home travel $t = \frac{d}{v_\mathrm{D}}$. Precisely, this way of reasoning is possible only in given limit $d \gg l$. The solution is now divided into two cases according to the sign of the argument of the absolute value.

Case when people overtake Doppler ($v_\mathrm{D} \leq v$)

Final frequency takes form

$$f_\mathrm{v} = \frac{v + v_\mathrm{D}}{l} + \frac{v - v_\mathrm{D}}{l} = 2\frac{v}{l}\,,$$

and is independent on Doppler's speed. The number of people encountered is therefore

$$N = f_\mathrm{v} t = 2\frac{v}{l}\frac{d}{v_\mathrm{D}}$$

with the minimal value if, and only if the speed is the maximum possible of the case $v_\mathrm{D} = v$. He totaly meets $N_\mathrm{min} = 2d/l$ people.

Case when Doppler overtakes people ($v_\mathrm{D} \geq v$)

We have the frequency

$$f_\mathrm{v} = \frac{v + v_\mathrm{D}}{l} + \frac{v_\mathrm{D} - v}{l} = 2\frac{v_\mathrm{D}}{l}\,.$$

Total number of the people he meets is given as

$$N = f_\mathrm{v} t = 2\frac{v_\mathrm{D}}{l}\frac{d}{v_\mathrm{D}} = 2\frac{d}{l} = N_\mathrm{min}\,,$$

so if he travels faster than v he encounters N_min people independent of his velocity. The answer to the task is therefore $N_\mathrm{min} = 2d/l = 2\,600$.

Note that interpreting the bywalkers as a wave with wavelength l and propagating at a speed v provides us with the frequency

$$f_\mathrm{p} = \frac{v + v_\mathrm{D}}{l}\frac{v}{v} = \frac{v + v_\mathrm{D}}{v} f_0\,,$$

where $f_0 = \frac{v}{l}$. Thus, this is just the Doppler effect for the moving observer.

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Problem 29 ... psoon

5 points

Jarda took a spoon out of a dishwasher and hung it on a cutlery stand. Obviously, Jarda does not hang cutlery perfectly, so the spoon is now swinging from side to side. Determine the period of its oscillations if we approximate its shape to be planar, composed of a circle of radius $R = 1.5\,\mathrm{cm}$ *connected to a rectangle of width* $a = 0.7\,\mathrm{cm}$ *and length* $b = 9\,\mathrm{cm}$. *The spoon is* $1\,\mathrm{mm}$ *thick and is made of a material with density* $\rho = 8000\,\mathrm{kg{\cdot}m^{-3}}$. *The spoon is hanging by a small hole which is located* $s = 1\,\mathrm{cm}$ *from the end of the spoon and in the middle of its width.*

Jarda lacks a dishwasher in the dorm.

We will use the relation for the period of a physical pendulum, which is

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the spoon with respect to the axis of rotation, m is its mass, g is the gravitational acceleration, and d is the distance of the center of gravity from the rotational axis. The center of gravity is located on the axis of symmetry of the spoon, so we only need to calculate one coordinate. We will measure the distance from the upper (rectangular) end of the spoon. The center of gravity of the rectangle is at $x_1 = \frac{b}{2}$, and the one of the circle is at $x_2 = b + R$. To find the position of the center of gravity of the entire object, we must use the relation

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}\,.$$

Because the thickness of the spoon is uniform, the mass of each part is proportional to its surface area. Thus, the position of the centre of gravity can be found as

$$x = \frac{ab\frac{b}{2} + \pi R^2\left(b + R\right)}{ab + \pi R^2} = 7.67\,\mathrm{cm}\,,$$

therefore, the distance of the centre of gravity from the axis of rotation is $d = x - s = 6.67\,\mathrm{cm}$. We still have to calculate the moment of inertia, which will be the sum of the moment of inertia of the circle and the rectangle with respect to the rotational axis. The moment of inertia of the circle with respect to the axis passing through its center is $\frac{1}{2}m_\mathrm{c}R_\mathrm{c}^2 = \frac{1}{2}\pi\rho tR_\mathrm{c}^4$, where m_c is the mass of the circle and R_c its radius. Using Steiner's parallel axis theorem, we find the moment of inertia of this part of the spoon with respect to the axis of rotation as

$$J_\mathrm{c} = \frac{1}{2}m_\mathrm{c}R_\mathrm{c}^2 + m_\mathrm{c}\left(b + R - s\right)^2 = \pi R^2 t\rho\left(\frac{R^2}{2} + \left(b + R - s\right)^2\right)\,,$$

$$J_\mathrm{c} \doteq 5.167\cdot 10^{-5}\,\mathrm{kg{\cdot}m^2}\,.$$

Similarly, we can find the moment of inertia of the rectangle with respect to the rotation axis (with respect to the center, it would be $\frac{1}{12}m_\mathrm{r}\left(a^2 + b^2\right)$) as

$$J_\mathrm{r} = \frac{1}{12}m_\mathrm{r}\left(a^2 + b^2\right) + m_\mathrm{r}\left(\frac{b}{2} - s\right)^2 = abt\rho\left(\frac{a^2 + b^2}{12} + \left(\frac{b}{2} - s\right)^2\right)\,,$$

$$J_\mathrm{r} \doteq 9.597\cdot 10^{-6}\,\mathrm{kg{\cdot}m^2}\,.$$

The mass of the entire spoon is

$$m = m_\mathrm{r} + m_\mathrm{c} = t\rho\left(ab + \pi R^2\right) ,$$
$$m \doteq 10.69\,\mathrm{g} .$$

The final equation representing the period is then

$$T = 2\pi\sqrt{\frac{\pi R^2\left(\frac{R^2}{2} + (b+R-s)^2\right) + ab\left(\frac{a^2+b^2}{12} + \left(\frac{b}{2} - s\right)^2\right)}{g\left(ab\frac{b}{2} + \pi R^2\,(b+R) - s\,(ab + \pi R^2)\right)}} ,$$
$$T \doteq 0.59\,\mathrm{s} .$$

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Problem 30 ... on a beach 6 points

Jarda is playing ball games with his friends on the beach. In water, Jarda can move with a speed of $0.7\,\mathrm{m{\cdot}s^{-1}}$*, while on land, he can move with a speed of* $1.0\,\mathrm{m{\cdot}s^{-1}}$*. Jarda is standing exactly on the straight line which forms the border between water and land and knows that the ball will fall near him in exactly* 3 *seconds. What is the area of the region (on land or water) where the ball might fall in such a way that Jarda will be able to catch it before it hits either land or water?* *Jarda spent some time on a beach in Greece instead of creating problems.*

Jarda has three options go. The first is to move on the beach, where in time $t = 3\,\mathrm{s}$ he can get to the distance $s_\mathrm{s} = v_\mathrm{s}t$ in any direction. On land, he can therefore cover an area of size

$$S_\mathrm{s} = \frac{\pi s_\mathrm{s}^2}{2} \doteq 14.14\,\mathrm{m}^2 .$$

The second option is to run in the water. Here the maximum distance is $s_\mathrm{v} = v_\mathrm{v}t$. However, there is a third, combined option. He can run on the beach, exactly at the borderline, and at some point begin to move through the water.

To describe this motion, we will use an analogy with light refraction. Light follows the curves with the shortest time; in other words, it travels the maximum possible distance in a given time. This is what we need. We will describe Jarda's motion using Snell's law, where the interface is the water-land borderline. However, Snell's law has some refractive indices that do not sunbathe at this beach, so we must change the law a little. By definition, the refractive index is the ratio of the speed of light in a vacuum c and the speed of light in a given medium v. We can then rewrite Snell's law as

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} .$$

In our scenario, Jarda is moving on the water-beach borderline, so α_1 is a right angle, and the angle at which he runs into the water (measured from the perpendicular to the shore) is equal to

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} .$$

No matter where Jarda decides to go into the water, it will always be best for him to head at this angle. Let's denote x the distance from the Jarda's initial position to where he decides to enter the water. The distance he can travel in water can be expressed as

$$s_x = v_\mathrm{v}\left(t - \frac{x}{v_s}\right) .$$

The boundary he can reach by this way is a straight line, with one end on the water-land borderline, distant 3.0 m from the initial position; and with the other end in the water, distant 2.1 m from the initial position, at the angle α_2 (measured from the perpendicular). The area that Jarda can cover in this way is, therefore, a right-angled triangle of area

$$S_\mathrm{k} = \frac{1}{2} v_\mathrm{v} t^2 \sqrt{v_\mathrm{s}^2 - v_\mathrm{v}^2} \doteq 2.25\,\mathrm{m}^2 .$$

Regarding the motion in the water, only a circle sector of the angle α_2 remains. It has an area of

$$S_\mathrm{v} = \frac{\alpha_2}{2\pi}\pi s_\mathrm{v}^2 \doteq 1.70\,\mathrm{m}^2 .$$

The last two areas we mentioned have to be considered twice due to the symmetry. In total, Jarda can cover the area of

$$S = S_\mathrm{s} + 2\left(S_\mathrm{k} + S_\mathrm{v}\right) = \frac{\pi\left(v_\mathrm{s} t\right)^2}{2} + v_\mathrm{v} t^2 \sqrt{v_\mathrm{s}^2 - v_\mathrm{v}^2} + \arcsin\frac{v_\mathrm{v}}{v_\mathrm{s}}\left(v_\mathrm{v} t\right)^2 \doteq 22.1\,\mathrm{m}^2 .$$

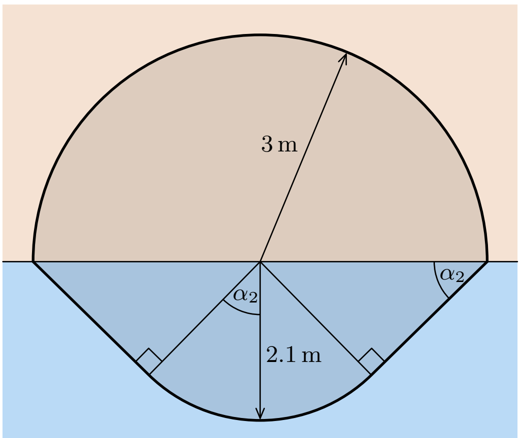

Fig. 4: Jarda's cross section.

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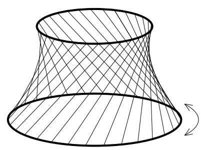
Problem 31 ... an oscillating hoop reloaded

6 points



We have two hoops. The upper one has a radius $r = 1.0\,\mathrm{m}$ and is fixed in a horizontal plane, and the bottom hoop has a radius $R = 1.5\,\mathrm{m}$. The bottom one is attached to the upper hoop by several massless cords, each of which has the same length, and if we look at the hoops from the top, we see the cords pointing radially. Therefore, the bottom hoop "lies" in the horizontal plane, too. The distance between the plane of the upper hoop and that of the bottom hoop is $h_0 = 2.0\,\mathrm{m}$. Let the mass of the bottom hoop be $m = 1.0\,\mathrm{kg}$. If we rotate the bottom hoop a bit around its vertical axis and release it, what is the period of its small oscillations?

Lego adjusted his problem from last year.

As is the common theme in physics, even the period of small oscillations can be calculated by multiple approaches. Here, we employ a perhaps less well known (you could have seen this approach in the solution to a similar problem in Physics Brawl Online 2020), but very efficient approach to the solution.

The approach is based on the law of conservation of energy. An appropriate coordinate for the description of the problem is the angle of rotation φ of the lower hoop relative to equilibrium.

Kinetic energy of the rotating thin loop is

$$E_\mathrm{k} = \frac{1}{2} I\omega^2 = \frac{1}{2} mR^2\dot{\varphi}^2 \,.$$

Description of potential energy is somewhat more involved. Lets focus now on a single cord and denote its length as l. Denoting the horizontal distance of points of attachement of the cord to the hoops as d and vertical distance of the said points as h, Pythagoras' theorem leads to $l^2 = h^2 + d^2$. In equilibrium, $d = R - r$ and $h = h_0$, and hence $l^2 = h_0^2 + (R-r)^2$. For the evaluation of the potential energy, the change of the distance between the planes the hoops lie in is critical.

We need to determine the distance d after rotating by angle φ. Since d is the horizontal distance, we can use a planar picture. Looking from above, the points of attachement of the cords to the hoops are colinear with the centre of the hoops in the equilibrium ($\varphi = 0$). After rotating by φ, the centre of the hoops, and the two points of attachement form a triangle with sides of lengths R, r, d, where sides R, r meet at angle φ. The distance d can be calculated using the law of cosines

$$d^2 = R^2 + r^2 - 2Rr\cos\varphi \,.$$

Pythagoras' theorem therefore dictates for the distance between the hoops

$$h^2 = l^2 - R^2 - r^2 + 2Rr\cos\varphi \,.$$

Substituting for l^2 and using that for $\varphi \ll 1$ we can approximate $\cos\varphi \approx 1 - \varphi^2/2$, we determine that

$$h^2 = h_0^2 + (R-r)^2 - R^2 - r^2 + 2Rr\left(1 - \frac{\varphi^2}{2}\right) = h_0^2 - Rr\varphi^2 \,.$$

We are however interested in the change of the distance of hoops compared to the equilibrium case. For small x, the approximation $(1+x)^a \approx 1 + ax$ holds. We therefore get

$$\Delta h = h_0\sqrt{1 - \frac{Rr\varphi^2}{h_0^2}} - h_0 \approx -\frac{Rr\varphi^2}{2h_0} \,.$$

Relative to the equilibrium position, the lower hoop was lifted by $-\Delta h$. The change in potential energy is then

$$E_\mathrm{p} = -mg\Delta h = \frac{1}{2}\frac{mgRr}{h_0}\varphi^2\,.$$

Remember that for a linear harmonic oscillator (point mass on a spring) the kinetic and potential energy are given as

$$E_\mathrm{k} = \frac{1}{2}m\dot{x}^2\,,$$
$$E_\mathrm{p} = \frac{1}{2}kx^2\,.$$

The period of oscillations is then $T = 2\pi\sqrt{m/k}$. We now only need to recognize that the relations for the energy of our system can be transformed to equations of LHO by defining the effective mass $m_\mathrm{ef} = mR^2$ and stiffness $k_\mathrm{ef} = \frac{mgRr}{h_0}$, since the period is independent of the choice of coordinates for the oscillator. The final result is therefore

$$T = 2\pi\sqrt{\frac{m_\mathrm{ef}}{k_\mathrm{ef}}} = 2\pi\sqrt{\frac{mR^2}{\frac{mgRr}{h_0}}} = 2\pi\sqrt{\frac{Rh_0}{rg}} \doteq 3.5\,\mathrm{s}\,.$$

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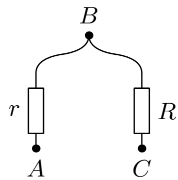
Problem 32 ... finite circuit — 6 points



In the figure, you can see a fragment of an electrical circuit with resistors $r = 2.35\,\Omega$ and $R = 271.2\,\Omega$. Any number of fragments can be connected together in a series according to several rules:

- *Each connector A (except for the first fragment) must be connected to exactly one B.*
- *Similarly, each B (except for the last fragment) must be connected to one A.*
- *All connectors C must be connected at one point.*
- *No other connections are allowed.*

Then we connect the A and C of the first fragment with one terminal of a multimeter, and we connect the B of the last fragment to the second terminal. What is the minimum number of fragments we need to use in order to measure a resistance greater than $R_x = 23.7\,\Omega$?

Jáchym wanted to come up with a problem about springs, but somehow it didn't work out.

Let's denote by R_n the resistance of n fragments. From the figure 5, we can derive

$$R_{n+1} = \frac{(r + R_n)\,R}{r + R_n + R}\,,$$

where R_0 is defined as $R_0 = 0$. We are looking for such a natural number n, for which $R_n \geq R_x$. We can write a simple script for this, but even a spreadsheet is sufficient. The result is $n = 26$.

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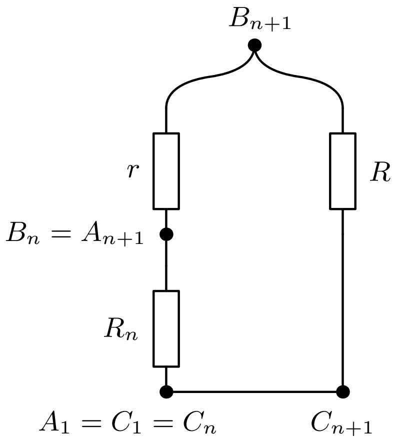

Fig. 5: Circuit diagram of the n+1-th fragment.

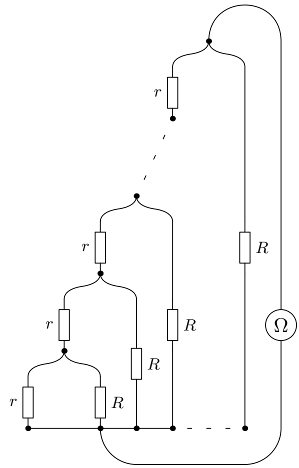

Fig. 6: Circuit diagram of the whole circuit.

Problem 33 ... an inclined roof

6 points

It is raining so much that $20\,\mathrm{mm}$ *of precipitation falls in an hour. The rectangular roof of an older building with dimensions* $a = 20\,\mathrm{m}$ *and* $b = 10\,\mathrm{m}$ *has a slope which is very small, but sufficient for all the water from the roof to flow into one place and fall to the ground from there. Determine the force which the stream of falling water exerts on the ground if it bounces upwards after impact with speed* $k = 5$*-times smaller than it was just before impact. Water drains from the roof so fast that a constant amount of water is maintained on it. The roof is at a height of* $h = 3\,\mathrm{m}$ *above the ground. Assume that the water draining from the roof has zero initial vertical velocity.*

Danka was walking in the rain around a building with an inclined roof.

The force by which water (falling from the roof and bouncing off the ground) acts on the ground can be determined using Newton's 2nd law as

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta m \Delta v}{\Delta t}\,.$$

Note that $\frac{\Delta m}{\Delta t}$ is the mass flow Q_m given by the mass of water that falls from the roof at any given moment, thus

$$Q_m = Rab\rho\,,$$

where $R = 20\,\mathrm{mm{\cdot}h^{-1}} = 5.55\,\mathrm{\mu m{\cdot}s^{-1}}$ and $\rho = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water. For the change of the water speed (the bouncing off case), the following holds

$$\Delta v = v_0 + \frac{v_0}{k} = v_0\left(1 + \frac{1}{k}\right)\,,$$

where the speed of a water flow v_0 is calculated from its free-fall as $v_0 = \sqrt{2gh}$, where we assume that the potential energy of water at height h has transformed into kinetic energy. This altogether yields

$$F = Rab\rho\sqrt{2gh}\left(1 + \frac{1}{k}\right) .$$

After substituting the given values, we find that the water flow acts on the ground with a force of $F \doteq 10.2\,\mathrm{N}$.

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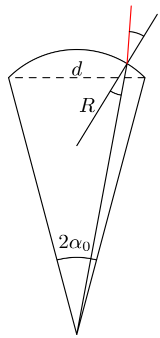
Problem 34 ... ommatidium

6 points



The eyes of insects are composed of many individual small units, the so-called ommatidia. We can consider each ommatidium to be a solid cone with vertex angle $2\alpha_0 = 14^\circ$. While photoreceptors are located at the vertex, the other side of the cone is enclosed by a spherical surface of radius $R = 17\,\mathrm{\mu m}$, which is coaxial with the cone. The diameter of the base of the cone is $d = 25\,\mathrm{\mu m}$. Determine the solid angle that one ommatidium can perceive if its interior is filled with a material of refractive index $n = 1.3$.

Jarda tried a school student project.

Let's begin in reverse order - consider the beam to be emitted from the photoreceptors to the space of the ommatidia. Then it refracts on a spherical surface towards the surrounding world. Let such a ray emanates at an angle α from the cone axis (the one which is perpendicular to bases). Then it impacts the spherical surface of an angle γ from the perpendicular (i.e., from the line connecting the center of curvature and the refractive point of the ray). From the sine theorem, we can determine the sine of the angle γ as

$$\sin\gamma = \frac{l}{R}\sin\alpha\,,$$

where l is the distance from the center of curvature (of ommatidium surface) to the apex of the cone, which can be determined as

$$l = \frac{d}{2\tan\alpha_0} - \sqrt{R^2 - \frac{d^2}{4}} = 90.3\,\mathrm{\mu m}\,.$$

According to Snell's law, we find the exit angle φ from the perpendicular to the spherical surface as

$$\sin\varphi = n\sin\gamma = n\frac{l}{R}\sin\alpha\,.$$

If we subtract angle φ from the angle between line (connecting the point of refraction at the spherical surface and the center of curvature) and the axis of the cone, we get the direction of the ray with respect to the axis of the cone. The angle of this direction can be expressed as

$$\beta = 180 - (180 - \gamma - \alpha) - \varphi = \gamma + \alpha - \varphi = \arcsin\left(\frac{l}{R}\sin\alpha\right) + \alpha - \arcsin\left(n\frac{l}{R}\sin\alpha\right).$$

For example, we can graphically verify that this function is negative and monotonically decreasing. It thus acquires extreme values at the edge of the ommatidia, that is when $\alpha = \alpha_0 = 7°$. For this value, the rays at angle $|\beta| = 10.0°$ from the ommatidia axis also get to the top of the cone. Because $|\beta| > \alpha_0$, the angle of view of the ommatidium is larger than its angular diameter. Note that due to the value of angle β being negative, the ommatidium sees everything in reverse to its axis. We determine the solid angle as the ratio of the area of the spherical cap with apex angle 2β, and the square of the radius of the sphere, i.e.

$$\Omega = \frac{2\pi R v}{R^2} = \frac{2\pi R^2 \left(1 - \cos\beta\right)}{R^2} = 2\pi \left(1 - \cos\beta\right) = 0.095\,\mathrm{sr}\,.$$

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Problem 35 ... thermodynamic cannon 6 points

We have a horizontally placed cylinder with a base area $S = 2.5\,\mathrm{dm}^2$ and a length $l = 2\,\mathrm{m}$. One end of the cylinder is closed and the other end is not. In the cylinder, there is a piston with a weight of $m = 24\,\mathrm{kg}$, which can move without friction and seals perfectly. The volume of the gas between the piston and the closed end is $V_0 = 3.5\,\ell$. First, everything is in equilibrium – the pressure outside the cylinder and also between the piston and the closed end is equal to normal atmospheric pressure and the temperature is T_0 everywhere. However, suddenly an explosion occurs – the temperature of the enclosed part of the gas changes to $10T_0$, and the amount of substance there becomes twenty times greater. At what speed does the piston leave the cylinder? Assume that classical equilibrium thermodynamics apply and the heat capacity ratio of the gas is $\kappa = 1.4$. *Lego wanted to play a joke on someone.*

Let's call the enclosed part of the gas (between the piston and the closed end of the cylinder) "the inside" and refer to all other gas as "the outside".

What will happen with the gas outside? We can assume the gas outside to be of infinite volume – if we compress it a little (by piston coming off the cylinder), the outside pressure should not change. Consequently, the piston will always be pressed back to the cylinder by pressure p_0.

What will happen with the gas inside? The explosion happened very quickly, therefore there is no time for heat exchange – we have to think of the explosion as an adiabatic process. The adiabatic process keeps the product pV^κ constant. Now we only need to compute the initial value of the pressure. The inside pressure for any position of the piston can be then calculated as

$$p_{\mathrm{in}} = p_{\mathrm{in}_0} \left(\frac{V_0}{V}\right)^\kappa ,$$

where particular volume V is calculated as the product of the area of the cylinder base and the distance between the edge of the piston and the closed base of the cylinder x. The initial value of this distance is $x_0 = V_0/S = 14\,\mathrm{cm}$. We can obtain the pressure just after the explosion using an equation of state $pV = nRT$. The temperature rises by a factor of ten, while the amount of the substance is twenty times its initial value, therefore the right-hand side increases by a factor of 200. The volume on the left-hand side is at the moment of explosion at its initial value, therefore the pressure has to increase by a factor of 200 $p_{\mathrm{in}_0} = 200 p_0$.

The resultant force acting on the piston is $F = S\left(p_{\mathrm{in}} - p_0\right)$. Someone may want to solve a differential equation; it is not necessary here – it is sufficient to integrate this force along the whole path to get the total work the gas did on the piston.

$$\begin{aligned} W &= \int_{x_0}^{l} S\left(200p_0\left(\frac{V_0}{Sx}\right)^{\kappa} - p_0\right)\mathrm{d}x = \left[Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}x^{1-\kappa} - x\right)\right]_{x_0}^{l} \\ &= Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}\left(l^{1-\kappa} - x_0^{1-\kappa}\right) - \left(l - x_0\right)\right). \end{aligned}$$

Now we only need to substitute for x_0. This work has to be equal to the kinetic energy of the piston

$$v = \sqrt{\frac{2}{m}W} = \sqrt{\frac{2}{m}Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}\left(l^{1-\kappa} - \left(\frac{V_0}{S}\right)^{1-\kappa}\right) - \left(l - \frac{V_0}{S}\right)\right)} \doteq 96\,\mathrm{m{\cdot}s^{-1}}\,.$$

If someone wanted to avoid the integration, one could obtain the exact solution by using the formula for the work of the adiabatic process (one can notice that even our solution is in the form of "the work done on the piston is equal to the difference of the work done by gas inside and outside").

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Problem 36 ... center of gravity of a snail 6 points

How far away from the center of a spiral is its center of gravity? Consider a spiral with constant linear density λ, given in polar coordinates as $r = a\varphi$, where $a = 0.1\,\mathrm{m{\cdot}rad^{-1}}$ and φ is the polar angle in radians. The total angle subtended by the spiral is 10π.

Jarda fights snails in his flowerbed.

We set up the Cartesian coordinates in the plane of the spiral. Let axis x have the direction of the angle $\varphi = 0$, and let the ray $\varphi = 90^\circ$ be the positive direction of axis y. Each point of the spiral can be determined by coordinates

$$\begin{aligned} x &= a\varphi\cos\varphi\,, \\ y &= a\varphi\sin\varphi\,. \end{aligned}$$

We can express the length of the element of the spiral arc $\mathrm{d}l$ dependent on angle φ as

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a\sqrt{\left(\cos\varphi - \varphi\sin\varphi\right)^2 + \left(\sin\varphi + \varphi\cos\varphi\right)^2}\,\mathrm{d}\varphi = a\sqrt{1+\varphi^2}\,\mathrm{d}\varphi\,,$$

where $\mathrm{d}x$ and $\mathrm{d}y$ are the differentials of the equations above. Now we can compute the length of the spiral by expression

$$L = a\int_0^{10\pi}\sqrt{1+\varphi^2}\,\mathrm{d}\varphi\,.$$

Let us firstly compute the x component of the location of the center of gravity. By the definition we get

$$x_C = \frac{\int_0^L x\lambda\,\mathrm{d}l}{\int_0^L \lambda\,\mathrm{d}l} = a\frac{\int_0^{10\pi}\varphi\cos\varphi\sqrt{1+\varphi^2}\,\mathrm{d}\varphi}{\int_0^{10\pi}\sqrt{1+\varphi^2}\,\mathrm{d}\varphi}\,.$$

Both integrals can be computed numerically, e.g., by using WolframAlpha (despite the existence of the analytical solution for the integral in the denominator). We get the first coordinate of the center of gravity

$$x_C \doteq \frac{6.264\,18\,\mathrm{m}}{495.80} \doteq 0.012\,634\,\mathrm{m}\,.$$

We compute the y coordinate of the center of gravity analogically as

$$y_C = a\frac{\int_0^{10\pi} \varphi \sin\varphi \sqrt{1+\varphi^2}\,\mathrm{d}\varphi}{\int_0^{10\pi} \sqrt{1+\varphi^2}\,\mathrm{d}\varphi} \doteq \frac{-98.708\,5\,\mathrm{m}}{495.80} \doteq -0.199\,09\,\mathrm{m}\,.$$

The center of gravity of this part of the spiral is distant

$$d = \sqrt{x_C^2 + y_C^2} = 0.199\,\mathrm{m}$$

from the origin. Regarding the size of the spiral (some points of it are as distant as 3 m from the origin), the location of the center of gravity is relatively close to the center of the spiral. Therefore the spiral is a pretty symmetrical object.

Because the spiral covers angle of 10π and most of its mass is located where φ acquires larger values, we can use approximation $\sqrt{1+\varphi^2} \doteq \varphi$. Integrals for the x coordinate of the center of gravity simplify to

$$x_C = a\frac{\int_0^{10\pi} \varphi^2 \cos\varphi d\varphi}{\int_0^{10\pi} \varphi\,\mathrm{d}\varphi} \doteq \frac{6.283\,\mathrm{m}}{493.5} \doteq 0.012\,7\,\mathrm{m}\,.$$

The other coordinate of the center of gravity returns $-0.199\,9\,\mathrm{m}$, which implies that the center of spiral and its center of gravity are distant $0.200\,\mathrm{m}$. Note that the result obtained by using approximation is very close to the exact one. However, the advantage is that the integrals have an analytical solution, and we do not have to compute them numerically.

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Problem 37 ... very thirsty

6 points

While wandering through a desert, Jarda came across a canister of water. The canister has a cuboid shape. It has width $c = 20\,\mathrm{cm}$, length $b = 40\,\mathrm{cm}$ and height $a = 60\,\mathrm{cm}$. The canister does not have a lid, but it has a tap through which water can flow out, in the middle of one of the bottom edges of length c. The container is 90 percent full. Jarda is so thirsty that he wants the water to flow from the tap as fast as possible. What is the angle (with respect to the vertical) by which he needs to tilt the canister around the edge c? Jarda first tilts the canister and then opens the tap. *Jarda was working at the steppes of South Moravia.*

The water flow rate is proportional to the square root of the height of the water column above the tap, so it is necessary to tilt the container so that the water level above the lower edge is as high as possible.

We find the dependence of the water level on the angle φ. Let h be the height of the water level above the tap, h_1 is the distance of the edge from the tap to the water level and h_2 is the distance from the edge of the base b to the water level. Obviously, we have to tilt the canister to the tap side, so $h_1 \geq h_2$.

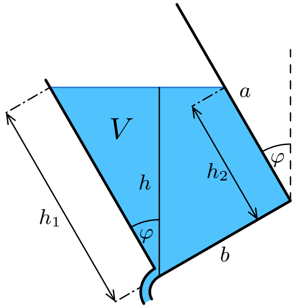

Fig. 7: The canister with water.

Assume that no water has leaked from the top of the canister. Then the volume of water is constant and equals $V = 0.9abc$. From a side view, this volume is an area S (bounded by the water surface and the sides of the container) multiplied by the length c. This area is $S = b\frac{h_1+h_2}{2}$. Notice that $h_1 \cos\varphi = h$ and $b\tan\varphi = h_1 - h_2$. Thus, we can write

$$h_1 = \frac{V}{bc} + \frac{b\tan\varphi}{2}$$

and

$$h = h_1\cos\varphi = \frac{V\cos\varphi}{bc} + \frac{b\sin\varphi}{2}\,.$$

Plugging the volume V into this equation provides

$$h = 0.9a\cos\varphi + \frac{b\sin\varphi}{2}\,.$$

We use the derivative to find the angle for which the function is at its maximum; this angle is $\varphi_{\max} = 20.3^\circ$. Because the canister has no lid, water starts leaking before the canister can be tilted by this angle. If some water leaks from the canister, the water level cannot be at its maximum anymore. The previous function is increasing on the interval from $\varphi = 0$ to $\varphi_{\max}$, so the height of the water level is maximal just when the level touches the upper edge of the canister. This occurs when $h_1 = a$. Therefore

$$\tan\varphi = \frac{2a - \frac{2V}{bc}}{b} = 2a\frac{0.1}{b}$$

and $\varphi = 16.7^\circ$.

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Problem 38 ... rotary capacitor 6 points

A capacitor consists of two rotary semicircular plates with the same radii $R = 2\,\mathrm{cm}$ at a mutual distance of $d = 0.1\,\mathrm{cm}$, separated by air. In the initial state, the plates are exactly on top of each other, the capacitor is charged to a voltage $U_0 = 20\,\mathrm{V}$ and then the voltage source is *disconnected. When rotating one plate away from the other (around the axis passing through the midpoint of the straight side of both plates), we act with a torque. What is the initial magnitude of this torque?* *Jarda couldn't loosen a screw.*

On the plates, the charge Q remains. According to the problem assignment, it is (air permittivity is approximately the same as for vacuum)

$$Q = CU_0 = \frac{\varepsilon S U_0}{d} = \frac{\varepsilon \pi R^2 U_0}{2d}.$$

When rotating the plates against each other, the capacitance of the capacitor changes and so does its energy $\frac{Q^2}{2C}$. At the beginning, the area of the capacitor is $\frac{\pi R^2}{2}$, when rotated by an angle φ the area of the capacitor is $\frac{(\pi-\varphi)R^2}{2}$. We substitute the formula for the capacitance $\frac{S\varepsilon}{d}$ into the expression for energy and differentiate with respect to angle, which gives us the torque (so-called virtual work principle)

$$\tau = \frac{\mathrm{d}E}{\mathrm{d}\varphi} = \frac{\partial E}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}\varphi} = \left(-\frac{Q^2}{2C^2}\right)\left(-\frac{\varepsilon R^2}{2d}\right).$$

The magnitude of the torque in the initial position $\varphi = 0$, when $C = \frac{\varepsilon \pi R^2}{2d}$, is

$$\tau = \frac{\varepsilon R^2 U_0^2}{4d} = 3.5 \cdot 10^{-10}\ \mathrm{N{\cdot}m}.$$

In the initial position, all marginal phenomena are negligible.

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Problem 39 ... inevitable fall 7 points

Imagine a body of a weight $M = 72\,\mathrm{kg}$, which we release from a height of $h = 12\,\mathrm{m}$ above the ground. At that moment, we start firing at it with a machine gun with a rate of fire $f = 2\,000\,\mathrm{min}^{-1}$ from the point where it would soon fall (the first projectile is shot at $1/f$ after releasing). The projectiles have a mass of $m = 11.3\,\mathrm{g}$, move at a speed of $v = 790\,\mathrm{m{\cdot}s^{-1}}$ and after collisions, they stay inside the body. How long would it take for the body to fall to the ground? *Jáchym knew it would eventually fall, but he wanted to know when exactly.*

Let's call the body's velocity immediately before i-th collision v_i and immediately after i-th collision v_i', both pointing down. The body is falling in a gravitational field, which means, that after time $T = f^{-1}$ it gains velocity $v_{i+1} - v_i' = gT$. Its mass will analogously be $M_i = M + (i-1)\,m$, $M_i' = M + im = M_{i+1}$. Now we write down the equation for perfectly inelastic collision, where only the momentum is conserved,

$$-mv + M_i v_i = (m + M_i)\,v_i' = M_{i+1} v_i',$$

where we've neglected all the small changes in projectiles' velocity due to gravity, for those changes would not (considering high velocity of projectiles and small value of T) have a chance to take effect. We as well neglect the fact, that the projectiles are in fact hitting the body with slightly different period than T, because the body's moving towards them. From the previous equation we get

$$\begin{aligned} v_i' &= \frac{-mv + M_i v_i}{M_{i+1}} = \frac{-mv + (M + (i-1)\,m)\,v_i}{M + im}\,, \\ v_{i+1} &= v_i' + gT = \frac{-mv + (M + (i-1)\,m)\,v_i + (M + im)\,gT}{M + im} = \\ &= \frac{-v + (k + i - 1)\,v_i + (k + i)\,gT}{k + i} \approx -\frac{v}{k + i} + v_i + gT\,, \end{aligned}$$

where $k = \frac{M}{m}$. The approximation at the end is reasonable, because $k \gg 1$ and so $(k + i - 1)\,/\,(k + i) \approx 1$. From here apparently

$$v_i \approx igT - v \sum_{j=2}^{i} \frac{1}{k + j}\,.$$

Notice, that we're computing the sum from $j = 2$, because $v_1 = gT$ has to hold – we're considering the first collision in the time T, so up until that moment, the body's falling "as usual".[3] It would now be easier for us to work with the velocity at the beginning of the interval after collision, so we express it by substituting for v_{i+1} into the original equation

$$v_i' = v_{i+1} - gT \approx (i+1)\,gT - v \sum_{j=2}^{i+1} \left(\frac{1}{k + j}\right) - gT = igT - v \sum_{j=1}^{i} \left(\frac{1}{k + j + 1}\right)\,.$$

Total distance traveled until the moment before n-th collision will thus be

$$\begin{aligned} x_n &= \sum_{i=0}^{n-1} \left(v_i' T + \frac{1}{2} gT^2\right) = gT^2 \sum_{i=0}^{n-1} i - vT \sum_{i=0}^{n-1} \sum_{j=1}^{i} \frac{1}{k + j + 1} + \frac{n}{2} gT^2 = \\ &= \frac{n^2}{2} gT^2 - vT \sum_{i=0}^{n-1} \sum_{j=1}^{i} \frac{1}{k + j + 1} = \frac{n^2}{2} gT^2 - vT\sigma(n)\,. \end{aligned} \tag{1}$$

Now we want to figure out, after how many steps will $x_n > h$ hold. To that end, we can use can use a spreadsheet software like MS Excel or some kind of script. The second option is to

[3] We're neglecting the time it takes the bullet to reach the body, so we assume, that the collision happens exactly at the same time as the shooting. The period $T \doteq 30\,\mathrm{ms}$ is of the same order as flight time of the first bullet (approximately 15 ms), so it might seem this assumption is unreasonable. However, the inaccuracy in the final result will truly be negligible.

approximate the last sum as

$$\begin{aligned}\sigma(n) &= \frac{1}{k}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left(1+\frac{j}{k}+\frac{1}{k}\right)^{-1} \approx \frac{1}{k}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left(1-\frac{j}{k}-\frac{1}{k}\right) = \\ &= \frac{1}{k^2}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left((k-1)-j\right) \quad = \frac{1}{k^2}\sum_{i=0}^{n-1}\left((k-1)\,i-\frac{i\,(i+1)}{2}\right) = \\ &= \frac{1}{2k^2}\sum_{i=0}^{n-1}\left((2k-3)\,i-i^2\right) \quad = \frac{1}{2k^2}\left((2k-3)\,\frac{n\,(n-1)}{2}-\frac{n\,(n-1)\,(2n-1)}{6}\right) = \\ &= \frac{n\,(n-1)\,(3k-4-n)}{6k^2}\,,\end{aligned}$$

which could be done, if $k \gg n$. This holds more than enough for given values values, which means that the function we've found approximates the sum almost with no inaccuracies. At the same time, it is in a way more interesting result than previously derived x_n, which describes the distance traveled by the body only at discrete times $t_n = nT$. Instead, we could now consider the continuous time t and define n outside of the integers as $n(t) = t/T$. By substituting into the expression (1) and by approximating the sum according to the equation above we get

$$X(t) = \frac{1}{2}gt^2 - \frac{vt}{6k^2}\left(\frac{t}{T}-1\right)\left(3k-4-\frac{t}{T}\right)\,.$$

We've already shown, that for integer values of n, $X(nT) \approx x_n$ holds. Now though, we'll go even further and without further proof we say, that $X(t)$ approximates the actual distance traveled by the body $x(t)$ not only in the "integer" times $t = nT$, but also for every real t.[4]

Some kind of physical intuition would be, that we're finding a 3rd polynomial to fit values x_n. And as we know, if the two polynomials of at most N-th degree are equal at at least $N+1$ points, they are necessarily equal at all the other points. And even though in this case functions x and X equal at the points x_n only approximately (and x apparently can't be a polynomial thanks to discontinuous first derivation), the approximation is correct enough with the required precision.

Now it only remains to let $X(t) = h$ and to find the time t at which body hits the ground. Situation is slightly complicated by the fact, that we're solving a general cubic in the form

$$0 = vt^3 + 3\left(gk^2T^2 - vT\,(k-1)\right)t^2 + vT^2\,(3k-4)\,t - 6hk^2T^2\,.$$

We can however easily numerically compute the only positive solution $t = 2.04\,\mathrm{s}$.

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[4] Obviously only for sufficiently small values – after shooting enough projectiles, the condition $k \gg n$ would cease to hold. Fortunately, the body would have fallen on the ground at this point.

Problem 40 ... life is short

7 points

Matěj found a piece of pure radium 225. After a while, he discovered that it decayed by β^- decay to actinium with a half-life of 15 days. However, the story does not end here. Actinium is subject to α decay to francium with a half-life of 10 days. In how many days (from the moment he found the radium) would Matěj have had the maximum amount of actinium?

Matěj stole this from a lecture on Computer Methods.

Let us denote the amount of radium that Matěj posesses at time t by $R(t)$, and the amount of actinium $A(t)$, respectively. Let λ_R and λ_A be the appropriate exponential decay constant. The setup can be described as follows

$$R(t) \xrightarrow{\lambda_R} A(t) \xrightarrow{\lambda_A} F(t)\,.$$

For $R(t)$ and $A(t)$, we can compile the following differential equations

$$\begin{aligned} \dot{R} &= -\lambda_R R(t)\ , \\ \dot{A} &= -\lambda_A A(t) + \lambda_R R(t)\ . \end{aligned} \tag{2}$$

The solution of the first equation is clearly

$$R(t) = R_0 \mathrm{e}^{-\lambda_R t}\,,$$

where R_0 is the initial amount of Matěj's radium. We plug this solution into (2)

$$\dot{A} = -\lambda_A A(t) + R_0 \lambda_R \mathrm{e}^{-\lambda_R t}\,.$$

The solution of this non-homogeneous differential equation is a bit more complicated.
By the initial condition $A(0) = 0$, we get the expression

$$A(t) = \frac{\lambda_R R_0}{\lambda_R - \lambda_A}\left(\mathrm{e}^{-\lambda_A t} - \mathrm{e}^{-\lambda_R t}\right)\,.$$

The amount of actinium is at its maximum if and only if the expression in brackets gets the extremal value. Now, from the equation $\dot{A}(t_{\mathrm{max}}) = 0$ we get

$$\lambda_A \mathrm{e}^{-\lambda_A t_{\mathrm{max}}} - \lambda_R \mathrm{e}^{-\lambda_R t_{\mathrm{max}}} = 0 \quad \Rightarrow \quad t_{\mathrm{max}} = \frac{t_R t_A}{t_A - t_R} \frac{\ln\left(\frac{t_A}{t_R}\right)}{\ln(2)} \doteq 17.5\,\mathrm{days}\,,$$

where $t_{R,A}$ represents half-lifes, we also used formula $\lambda_{R,A} = \frac{\ln(2)}{t_{R,A}}$.

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Problem 41 ... radiant

5 points

An ideal black body radiates with the maximum intensity I_λ at a wavelength of $\lambda_{\mathrm{max}} = 598.34 \cdot 10^{-9}\,\mathrm{m}$. *At what wavelength does the black body radiate with half of the maximum intensity? Submit the largest correct answer.*

Danka thought she wouldn't like to solve this problem.

The black-body radiation is described by the Planck's law. The law can be written in two forms

$$\mathrm{d}I_\nu = B_\nu \,\mathrm{d}\nu\,, \qquad \mathrm{d}I_\lambda = B_\lambda \,\mathrm{d}\lambda\,,$$

where quantities with the index ν differ from quantities marked with the index λ. They are coupled together, and we can obtain one from another respecting the condition of total intensity equality

$$I = \int_0^\infty B_\nu \,\mathrm{d}\nu = \int_0^\infty B_\lambda \,\mathrm{d}\lambda\,.$$

To calculate it, we need the relation between the wavelength λ and the frequency ν

$$\nu\lambda = c \rightarrow \mathrm{d}\nu = -\frac{c}{\lambda^2}\,\mathrm{d}\lambda\,.$$

A well-known form of the Planck's law for B_ν gives us

$$B_\lambda = B_\nu(\nu = \frac{c}{\lambda}) \cdot \frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}\,.$$

We can obtain the function's maximum by differentiating it; however, it does not have a nice analytic solution (it contains the Lambert W function). The final result is called Wien's displacement law, and it holds as

$$\lambda_{\mathrm{max}} T = b = 2.897771955 \cdot 10^{-3}\,\mathrm{m{\cdot}K}\,.$$

Using the given maximum wavelength value, we get the temperature of the black body $T = 4843.0\,\mathrm{K}$. The next step is to determine the value of the radiation intensity in the maximum. By plugging into we get $B_\lambda^{\mathrm{max}} = 1.09122 \cdot 10^7\,\mathrm{W{\cdot}m^{-2}{\cdot}sr^{-1}{\cdot}\mu m^{-1}}$.

The most challenging part of the problem is to determine when

$$B_\lambda(\lambda) = \frac{1}{2}B_\lambda^{\mathrm{max}} = 5.4561 \cdot 10^6\,\mathrm{W{\cdot}m^{-2}{\cdot}sr^{-1}{\cdot}\mu m^{-1}}\,.$$

We already know all necessary quantities so that we can solve the equation numerically. A different approach is to plot Planck's function for a given temperature and determine the solution from the graph. There are several helpful web tools for dealing with Planck's law, such as `https://www.spectralcalc.com/blackbody_calculator/blackbody.php`, which plots the dependence on the wavelength interval. By the graph of the whole function, we can evaluate that the half value for greater wavelengths is located somewhere in the $1000-1200\,\mathrm{nm}$ interval. After plotting the interval, we can narrow it down to $1080 - 1090\,\mathrm{nm}$ and then to $1087 - 1088\,\mathrm{nm}$. Finally, it can be estimated that $\lambda_{1/2} = 1087.2\,\mathrm{nm}$.

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Problem 42 ... anti-reflective glasses

7 points

Patrik has a thin anti-reflective layer on his glasses of a thickness $d = 250\,\mathrm{nm}$ and of a refractive index $n_2 = 1.4$. Which part of the visible light is subject to destructive interference? The refractive index of air is $n_1 = 1$, for glass it is $n_3 = 1.55$ and light rays are perpendicular to the glasses. *Patrik was about to buy sunglasses.*

Since $n_1 < n_2 < n_3$ applies to individual layers, the light always changes phase by $\frac{\lambda}{2}$ if reflected on a more dense layer, and because it happens twice, the chages cancel out each other. The condition for interference the minimum is $\Delta l = (2k+1)\frac{\lambda}{2}$, while the difference of the optical path when passing through the reflective layer is $\Delta l = 2n_2 d$. These two paths have be equal

$$2n_2 d = (2k+1)\frac{\lambda}{2},$$
$$\lambda = \frac{4n_2 d}{2k+1}.$$

We are looking for a multiple of k, for which the part of the visible spectrum $(400\,\mathrm{nm}, 700\,\mathrm{nm})$ is canceled out

$$\lambda(k=0) = \frac{4n_2 d}{1} = \frac{4 \cdot 1.4 \cdot 250}{1}\,\mathrm{nm} = 1\,400\,\mathrm{nm},$$
$$\lambda(k=1) = \frac{4n_2 d}{3} = \frac{4 \cdot 1.4 \cdot 250}{3}\,\mathrm{nm} = 467\,\mathrm{nm},$$
$$\lambda(k=2) = \frac{4n_2 d}{5} = \frac{4 \cdot 1.4 \cdot 250}{5}\,\mathrm{nm} = 280\,\mathrm{nm}.$$

Hence, the light of a wavelength of 467 nm, which corresponds to the blue part of the spectrum, is subject to destructive interference.

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Problem 43 ... Dano is flying to an exoplanet — 7 points

For exoplanet research, Dano would like to fly to Proxima Centauri, at a distance of 4.2 light-years. Calculate the Lorentz factor if Dano wants to get exactly half a year older on his way there and back. Do not consider acceleration of the spaceship.

Vašek was jealous of Dano's research on exoplanets.

Due to a time dilatation, Dano will age a total of

$$\Delta\tau = \frac{\Delta t}{\gamma}$$

where Δt is the time elapsed on Earth and γ is the Lorentz factor. According to the values from the assignment, it is clear that Dano will have to travel almost at the speed of light c; therefore, we can estimate

$$\Delta t \approx \frac{2l}{c}\,,$$

where $2l = 8.4\,\mathrm{ly}$. By combining the two equations above we obtain

$$\gamma \approx \frac{2l}{c\Delta\tau} \doteq 16.8\,.$$

In a more accurate calculation, we would assume that Dano moves at a speed of v with respect to the Solar System. Then for observers on Earth, Dano's journey will take time

$$\Delta t = \frac{2l}{v}\,.$$

Again, by combining the two equations we exculde the time Δt,

$$\frac{\Delta\tau}{2l} = \frac{1}{\gamma v} = \frac{1}{v}\sqrt{1-\frac{v^2}{c^2}}$$

From here, by simple algerbraic adjustements, we can express the factor γ as

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{1+\left(\frac{2l}{c\Delta\tau}\right)^2} \doteq 16.8\,.$$

Within the accuracy of the problem, we received the same result as above.

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Problem 44 ... platform 9 and 3/4 — 6 points

We are standing on a train platform 2 m *away from the track. Next to us, there is a very long train which is slowly braking. We are located roughly near the middle of the train. We hear a really unpleasant squeal of brakes with loudness of* 110 dB *coming from the entirety of the train. Moving away from the track might help us a bit. How many decibels do we measure at the other side of the platform,* 7 m *away from the track?*

Jarda believes in Harry Potter and loves Hermione.

Firstly, we need to know how does the sound intensity I changes with a distance. The intensity of the sound is defined as an acoustic power carried by sound waves through a specific area. Because the sound comes out of the whole very long train, we can consider this area to be a lateral surface of a cylinder with the axis being in the center of the train. The surface area grows linearly with distance from the train; the intensity is thus inversely proportional to the distance – this fact is crucial to understand the problem; unlike point source, where $I \propto 1/r^2$, in this case, it holds $I \propto 1/r$.

The sound loudness L (in decibels) is given by

$$L = 10\log_{10}\frac{I}{I_\mathrm{p}}\,,$$

where I_p is the intensity of the threshold of perception.

Let's try to subtract the loudnesses at distances $r_1 = 2\,\mathrm{m}$ and $r_2 = 7\,\mathrm{m}$ from the train

$$L_1 - L_2 = 10\left(\log_{10}\frac{I_0}{r_1 I_\mathrm{p}} - \log_{10}\frac{I_0}{r_2 I_\mathrm{p}}\right) = 10\log_{10}\frac{r_2}{r_1}\,.$$

Now we can easily express the loudness L_2 as

$$L_2 = L_1 - 10\log_{10}\frac{r_2}{r_1} = 104.6\,\mathrm{dB}\,.$$

The intensity has changed quite a bit; however, our perception did not change that much.

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Problem 45 ... a ride on the exponential 8 points

The exponential is an essential function in physics, which is why Jarda built a roller coaster in the shape of this function in his new amusement park. The ride starts at a height of $h = 10\,\mathrm{m}$. *The cart then goes down a path which, if viewed from the side, has a profile given by the function* $y = h\mathrm{e}^{-\frac{x}{h}}$, *where* x *is the distance in the horizontal direction from the base of the entrance tower at the beginning of the ride. At what height above the ground does the descending cart exert the greatest force on the roller coaster?*

Jarda is tired of taking the elevator.

On the descent, the cart with the visitors exerts two forces on the rollercoaster – gravitational and centripetal. The centripetal force acts perpendicularly to the track, and the gravitational force has to be decomposed into a normal and a tangent component. The resulting force will be the sum of the normal gravitational force and the centripetal force. The magnitude of the centripetal force can be found using the speed of the cart and the radius of curvature. First, we will determinate the speed v using the law of conservation of energy

$$v = \sqrt{2g\left(h - y(x)\right)} = \sqrt{2gh(1 - \mathrm{e}^{-\frac{x}{h}})}\,,$$

where g is the gravitational acceleration and $y(x)$ is the height above the ground depending on the horizontal coordinate. The radius of curvature r can be found using the formula

$$r = \frac{(1 + {y'}^2)^{\frac{3}{2}}}{|y''|}\,,$$

where y', y'' are the first and the second derivative of the function y with respect to x. The exponential is relatively easy to differentiate, so we can write the expression for the centripetal force with respect to x right away as

$$F_\mathrm{c} = \frac{mv^2}{r} = mg\frac{2\left(1 - \mathrm{e}^{-\frac{x}{h}}\right)\mathrm{e}^{-\frac{x}{h}}}{\left(1 + \left(-\mathrm{e}^{-\frac{x}{h}}\right)^2\right)^{\frac{3}{2}}}\,.$$

Now we can find the normal component of the gravitational force

$$F_{g_\mathrm{n}} = mg\cos\alpha = mg\frac{1}{\sqrt{1 + \tan\alpha^2}}\,,$$

where α is the angle between the slope of the track and the ground. In a given point, the function $\tan\alpha$ is numerically equal to the derivative of y with respect to x, therefore we can write

$$F_{g_\mathrm{n}} = mg\frac{1}{\sqrt{1 + \mathrm{e}^{-\frac{2x}{h}}}}\,.$$

Both the centripetal and gravitational force act on the track, which is why we can add them up to find the resulting force that the cart exerts on the rollercoaster

$$F = mg\left(\frac{1}{\sqrt{1 + \mathrm{e}^{-\frac{2x}{h}}}} + \frac{2\left(1 - \mathrm{e}^{-\frac{x}{h}}\right)\mathrm{e}^{-\frac{x}{h}}}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^{\frac{3}{2}}}\right) = mg\frac{1 + 2\mathrm{e}^{-\frac{x}{h}} - \mathrm{e}^{-\frac{2x}{h}}}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^{\frac{3}{2}}}\,.$$

To find the greatest value of this function, we will differentiate it with respect to x

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{\mathrm{e}^{-\frac{3x}{h}} - 4\mathrm{e}^{-\frac{2x}{h}} - 5\mathrm{e}^{-\frac{x}{h}} + 2}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^2} \,.$$

To find the extremum, we equalize the derivative to zero

$$\mathrm{e}^{-\frac{3x}{h}} - 4\mathrm{e}^{-\frac{2x}{h}} - 5\mathrm{e}^{-\frac{x}{h}} + 2 = 0$$

and solve the equation numerically. We obtain that the only positive numerical solution is 11.30 m, where the track is 3.23 m above ground. Thus, at this height, the descending cart will exert the greatest force on Jarda's rollercoaster.

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Problem 46 ... carousel ride — 7 points

In addition to the roller coaster, Jarda also bought a chain carousel for his new amusement park. A horizontal disc with a diameter of $d = 8\,\mathrm{m}$ rotates with a period of $T = 8\,\mathrm{s}$, and the seats are attached to the edge of the disc using ropes of length $l = 8\,\mathrm{m}$. Determine the force exerted by a child weighing $m = 25\,\mathrm{kg}$ on a seat when the carousel is in a steady state.

Jarda works around the clock.

In a rotating coordinate system, a child is subject to three forces - tensile force of the rope, the force of gravity, and centrifugal force. In a steady state, the net force and torque are zero. Let us denote the angle between the rope and perpendicular to the ground as α. Then the seat is distant $x = \frac{d}{2} + l\sin\alpha$ from the rotational axis, and the magnitude of centrifugal force is

$$F_\mathrm{c} = m\omega^2 x = m\frac{4\pi^2}{T^2}\left(\frac{d}{2} + l\sin\alpha\right) \,.$$

The resultant force that the child is acting on the seat is the sum of gravity and centrifugal force. Thus, the magnitude of the resultant force is

$$F = \sqrt{F_g^2 + F_\mathrm{c}^2} \,.$$

From the equality of the moments of the two forces, it holds

$$\tan\alpha = \frac{F_\mathrm{c}}{F_g} \,.$$

Now we substitute from the last two equations into the first one and get

$$F_\mathrm{c} = \frac{4\pi^2}{T^2}m\left(\frac{d}{2} + l\frac{F_\mathrm{c}}{F}\right) \,,$$

$$F_\mathrm{c}\left(1 - \frac{4\pi^2 ml}{FT^2}\right) = \frac{4\pi^2}{2T^2}md \,,$$

$$\left(F^2 - m^2g^2\right)\left(F - \frac{4\pi^2 ml}{T^2}\right)^2 = \frac{4\pi^4 m^2 d^2}{T^4}F^2 \,.$$

This is the equation of degree four in variable F; therefore, we must solve it numerically. The only positive root is the sought value $F = 270\,\mathrm{N}$.

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Problem 47 ... catapult

7 points

We thread a small bead on a rod of a length $l = 30\,\mathrm{cm}$, which lies on a horizontal pad and can only rotate vertically around one of its ends, which is fixed on the pad. We start to rotate the rod upwards at a constant angular speed. What is the maximum distance from the center of rotation at which we can place the bead such that it leaves the rod just at the moment when it reaches the vertical position? The bead moves along the rod without friction.

Jarda wanted to hit the professor while in a lecture on Theoretical Mechanics.

Let's solve the problem in the reference frame associated with the rotating rod. In such a frame, gravitational, centrifugal, Coriolis and rod reaction forces will act on the bead. However, since the bead can only move radially relative to the rod, the Coriolis force will have no effect on the motion, because it will cancel with the reaction force component of the rod. Let us describe the position of the bead by the coordinate x, which indicates its distance from the centre of rotation. The centrifugal force has a magnitude of

$$F_{\mathrm{ods}} = m\omega^2 x\,,$$

where m is mass of bead and ω is the angular velocity of rotation of the rod. This force acts in the direction away from the centre of rotation. Another important force is the gravitational force. Let's divide it into two components, one in the direction of the rod and the other perpendicular to it. We can calculate the components in the direction of the rod as

$$F_{\mathrm{gt}} = mg\sin\varphi = mg\sin\omega t\,,$$

where g is gravitational acceleration and $\varphi = \omega t$ is angle of inclination of the rod with respect to the horizontal plane. As the angle of inclination increases with time, this component of the gravitational force acts towards the centre of rotation. As with Coriolis force, the normal component of the gravitational force will be compensated by the reaction of the rod, because the bead only moves radially in our reference frame. The equation of motion of the bead on the rod is

$$ma = m\omega^2 x - mg\sin\omega t\,.$$

This is a second order differential equation. However, we can try to guess its solution. We divide the function x into two parts, x_1 and x_2. We are looking for a function x_1, which, if we derive twice, we get the same function multiplied by the factor ω^2. The function satisfying part of the condition is the sum of two exponentials, i.e.

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 A e^{Bt} + C e^{Dt}}{\mathrm{d}t^2} = B^2 A e^{Bt} + D^2 C e^{Dt}\,.$$

Now we apply the condition $\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \omega^2 x$ and we obtain the relation

$$B^2 A e^{Bt} + D^2 C e^{Dt} = \omega^2\left(A e^{Bt} + C e^{Dt}\right)\,.$$

For the two sides to be equal, the constants B and D must be equal to $B = \omega$ and $D = -\omega$ (or the opposite, if they were both the same, then it would be just one exponential, and it can't be any other numbers because of the ω^2 factor). We derive the factors A and C from the initial conditions.

In a similar way we can find the second part of the function x, x_2. This is the part whose second derivative is equal to this function multiplied by the ω^2 factor minus the sine function. So let's try to put $x_2 = E\sin\omega t$. Then

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 E\sin\omega t}{\mathrm{d}t^2} = -E\omega^2 \sin\omega t\,.$$

We know that

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -E\omega^2 \sin\omega t = \omega^2 x_2 - g\sin\omega t = \omega^2 E\sin\omega t - g\sin\omega t\,.$$

In order for this equation to be true, it is necessary that

$$E = \frac{g}{2\omega^2}\,.$$

Thus

$$x = Ae^{\omega t} + Ce^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\,.$$

The first initial condition is zero velocity at time $t = 0$. Let's derive the function x to find the velocity of the bead v and after putting $t = 0$ we get the equation

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega e^{\omega t} - C\omega e^{-\omega t} + \frac{g}{2\omega}\cos\omega t = 0\,,$$

from which we get

$$A - C = -\frac{g}{2\omega^2}\,.$$

According to the task, at an angle of $\varphi = \omega t = \frac{\pi}{2}$ the bead should fly out of the rod, so it must be true that

$$l = Ae^{\frac{\pi}{2}} + Ce^{-\frac{\pi}{2}} + \frac{g}{2\omega^2}\,,$$

from which, using the first condition, we can derive

$$A = \frac{l - \frac{g}{2\omega^2}\left(1 + e^{-\frac{\pi}{2}}\right)}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}} = \frac{l - \frac{g}{2\omega^2}\left(1 + e^{-\frac{\pi}{2}}\right)}{2\cosh\frac{\pi}{2}}\,.$$

Therefore

$$C = A + \frac{g}{2\omega^2} = \frac{l - \frac{g}{2\omega^2}\left(1 - e^{\frac{\pi}{2}}\right)}{2\cosh\frac{\pi}{2}}\,.$$

At time $t = 0$, the position of the bead is

$$x_0 = A + C = \frac{l - \frac{g}{2\omega^2} - \frac{g}{4\omega^2}\left(e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}\right)}{\cosh\frac{\pi}{2}}\,.$$

The greater the angular velocity, the higher the initial position. For $\omega \to \infty$ the value of the initial distance will be

$$x_0 = \frac{l}{\cosh\frac{\pi}{2}} = 12.0\,\mathrm{cm}\,.$$

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Problem 48 ... too many cubes
5 points

Consider an infinite cube net consisting of identical resistors $R = 24\,\Omega$, which are located on the edges of all cubes. What resistance do we measure between two adjacent vertices?

Jarda wanted a challenging problem with a short solution.

Let us bring the current I to vertex A. Considering the symmetry of the net, the current flowing to each of the six adjacent vertices is $\frac{I}{6}$. From the adjacent vertex B, the current of the same magnitude I has to flow away and from the symmetry again, each current flowing into the B is equal to $\frac{I}{6}$. From the superposition principle, we get that the current flowing between vertices A and B is

$$\frac{I}{6} + \frac{I}{6} = \frac{I}{3}\,,$$

thus, the voltage on the resistor between A and B is $U = R\frac{I}{3}$. This voltage is equal to the product of the current I and the resulting resistance R_c of the cube net between vertices A and B. Therefore, $U = R_\mathrm{c} I$, and thus,

$$R_\mathrm{c} = \frac{U}{I} = \frac{R\frac{I}{3}}{I} = \frac{R}{3} = 8\,\Omega\,.$$

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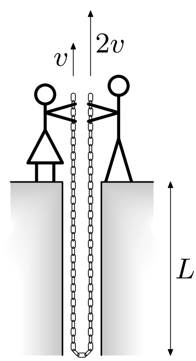
Problem 49 ... we work by pulling
8 points

Alice and Bob are pulling a non-elastic chain of linear density $\lambda = 0.4\,\mathrm{kg{\cdot}m^{-1}}$ from a pit. Alice is pulling the chain out at a constant speed $v = 0.6\,\mathrm{m{\cdot}s^{-1}}$ and Bob at twice the speed. Consider the work done between the point in time when the lowest point of the chain is at a depth of $L = 5\,\mathrm{m}$ and when it is fully pulled out. How many times larger will Bob's work be than Alice's? Assume that they have already been pulling for some time at the starting point.

Matěj is fascinated by the Mould effect.



At a first glance, we might think that Bob needs to do double the work, because both Bob and Alice pull by the same force and Bob pulls on double the distance. However, he needs to pull by a larger force, because as he is pulling, he accelerates the chain from the Alice's velocity $v_\mathrm{A} = v$ to $v_\mathrm{B} = 2v_\mathrm{A}$. We can assume that Alice and Bob are close to each other relative to the chain's length, so the lower bent part of the chain is negligibly small.

Let a depth of the lowest chain's point be l, in the beginning $l = L$ and in the end $l = 0$. Alice exerts a force on the chain, which is just as large as gravity acting on Alice's half of the chain,

$$F_\mathrm{A} = F_g = \lambda l g\,.$$

We need to use a general Newton's motion law to evaluate Bob's force. A resultant force acting on his part of the chain must be equal to the time derivative of a momentum

$$F_\mathrm{B} - F_g = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}\,(mv)}{\mathrm{d}t} = v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t}\,.$$

Bob's part of the chain does not accelerate as a whole, therefore the last term equals zero. The first term tells us how large a force needs to be to accelerate a mass $\mathrm{d}m$ during the time $\mathrm{d}t$ by a velocity difference Δv. In this case $\Delta v = v_\mathrm{B} - v_\mathrm{A}$.

During the period $\mathrm{d}t$ there will be pulled out $\mathrm{d}x = (v_\mathrm{A} + v_\mathrm{B})\,\mathrm{d}t$ of the chain. Just before this moment, one half of this length has been on Alice's side and the other half on Bob's side. On the other hand, Bob pulled out a different chain's length $v_\mathrm{B}\,\mathrm{d}t$ and therefore a segment "flowed" to his part with a length

$$\mathrm{d}l_\mathrm{B} = v_\mathrm{B}\,\mathrm{d}t - \frac{\mathrm{d}x}{2} = \frac{v_\mathrm{B} - v_\mathrm{A}}{2}\,\mathrm{d}t\,.$$

A weight of this segment is $\mathrm{d}m = \lambda\,\mathrm{d}l_\mathrm{B}$ and we obtain the Bob's total force by plugging values in

$$F_\mathrm{B} = F_g + v\frac{\mathrm{d}m}{\mathrm{d}t} = F_g + \frac{\lambda}{2}\left(v_\mathrm{B} - v_\mathrm{A}\right)^2\,.$$

Why is this additional force exerted only by Bob and not both as with gravity? The answer is that only Bob causes the acceleration. Alice's force effects are equivalent to a situation where she would pull her part with a length l, from whose end the chain's link were falling off. It would shorten itself, but no force is created. In fact, these links will be pulled to Bob, so he needs some extra force.

We find the lowest point of the chain at the time t as

$$l = L - \frac{v_\mathrm{A} + v_\mathrm{B}}{2}t\,,$$

The total pulling time will be $T = \frac{2L}{v_\mathrm{A}+v_\mathrm{B}}$. Now we have one remaining job: solve for work. For Alice's work, we get

$$W_\mathrm{A} = \int_0^T F_\mathrm{A} v_\mathrm{A}\,\mathrm{d}t = \lambda g v_\mathrm{A} \int_0^T \left(L - \frac{v_\mathrm{A}+v_\mathrm{B}}{2}t\right)\mathrm{d}t = \lambda g v_\mathrm{A}\left[Lt - \frac{v_\mathrm{A}+v_\mathrm{B}}{4}t^2\right]_0^T = \\ = \frac{\lambda g v_\mathrm{A} L^2}{v_\mathrm{A}+v_\mathrm{B}} = \frac{\lambda g L^2}{3}\,.$$

This applies for Bob as well. The contribution of gravity will be almost the same, just twice as large due to twice the speed. So let's focus on the second force

$$W_\mathrm{B} = \int_0^T F_\mathrm{B} v_\mathrm{B}\,\mathrm{d}t = \frac{v_\mathrm{B}}{v_\mathrm{A}}W_\mathrm{A} + \int_0^T \frac{\lambda}{2}\left(v_\mathrm{B}-v_\mathrm{A}\right)^2 v_\mathrm{B}\,\mathrm{d}t = 2W_\mathrm{A} + \lambda L\frac{\left(v_\mathrm{B}-v_\mathrm{A}\right)^2}{v_\mathrm{B}+v_\mathrm{A}}v_\mathrm{B} = \\ = 2W_\mathrm{A} + \frac{2\lambda L v_\mathrm{A}^2}{3}\,.$$

To finish, we have the works' ratio

$$\frac{W_\mathrm{B}}{W_\mathrm{A}} = 2 + \frac{2\lambda L v_\mathrm{A}^2}{3W_\mathrm{A}} = 2 + \frac{2v_\mathrm{A}^2}{gL} = 2.015\,.$$

You may find interesting that the ratio doesn't depend on chain's length density, but does depend on gravity, the initial depth and the pulling speed.

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Problem 50 ... snowball to a window

8 points

Jarda wanted to distract his quarantined friend, so he decided to throw a snowball at his window. The lower frame of the window is at a height of $h_1 = 3.5\,\mathrm{m}$ and the upper frame at $h_2 = 4.8\,\mathrm{m}$. The window is $d = 2\,\mathrm{m}$ wide. Jarda was standing four meters from the base of the house, right in front of the center of the window. What part of the area of the window (in percent) could he hit if he threw the ball at a speed of $v = 8\,\mathrm{m{\cdot}s^{-1}}$, from $H = 1.8\,\mathrm{m}$ above ground?

Jarda aims high.

We introduce the Cartesian coordinate system with its origin at the place where Jarda is standing so that the plane xy represents the ground and the y-axis points to the horizontal center of the window. Jarda can throw the snowball only at a particular area of the space, which is delimited by what is called a parabola of safety (in our three-dimensional space, a rotational paraboloid). We will not derive the equation of this paraboloid here, but the height z of the protective paraboloid with respect to the distance r from Jarda's feet is

$$z = H + \frac{v^2}{2g} - \frac{r^2 g}{2v^2}\,.$$

Points, where the paraboloid intersects the window determine the border of all points Jarda can hit. These points lie at a ground distance of

$$r = \sqrt{D^2 + x^2}$$

from the place where Jarda is standing. The distance between Jarda and the window is $D = 4\,\mathrm{m}$, and x is the position of a point from the center of the window. Thus, the safety paraboloid intersects the window at points

$$z = H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{x^2 g}{2v^2}\,. \tag{3}$$

The area under these points can be easily calculatad with the integral

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \left(H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{(x^2) g}{2v^2}\right) \mathrm{d}x = \left[Hx + \frac{v^2}{2g}x - \frac{D^2 g}{2v^2}x - \frac{x^3 g}{6v^2}\right]_{-\frac{d}{2}}^{\frac{d}{2}} =$$

$$= \left(H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2}\right) d - \frac{d^3 g}{24v^2}\,.$$

We will now subtract the area of the wall under the window - that is $h_1 d$, and in order to obtain the searched ratio, we devide this value by the total area of the window $(h_2 - h_1)\,d$. Thus, the solution is

$$\frac{H - h_1 + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{d^2 g}{24v^2}}{(h_2 - h_1)} = 23.9\,\%\,.$$

It is also important to verify that our area does not extend above the window. We determine the maximum height from the equation (3) by substituting $x = 0$. After numerical substitution we get $z \doteq 3.8\,\mathrm{m}$, which is less than h_1, so we can't hit the area above the top edge of the window.

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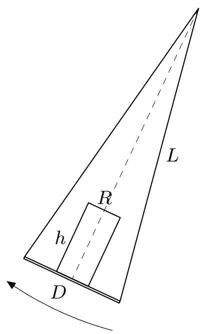
Problem 51 ... a bottle on a swing

8 points



Jarda is swinging harmoniously on a swing with a frequency $f = 0.3\,\mathrm{Hz}$. He has a full cylinder-shaped bottle with him, placed exactly in the middle of the width of the swing. The bottle has a height $h = 30\,\mathrm{cm}$ and it stands on its base of a diameter $R = 3.0\,\mathrm{cm}$. The swing is $D = 40\,\mathrm{cm}$ wide and hangs on ropes that are $L = 1.4\,\mathrm{m}$ long. The friction between the swing and the bottle is enormous, so the bottle stands in its place. Determine the smallest amplitude of the swing's inclination angle upon which the bottle starts to wobble. *Jarda keeps dropping things.*

Let us consider a system connected to a swing. In this system, the forces acting upon the bottle are gravitational, a reaction of the swing, inertial, friction and centrifugal. We will decompose the forces in directions perpendicular and parallel to the swing and we will examine their torques. Friction is acting upon the bottle in the direction that is tangential to the swing and its point of action lies in the plane of the lower base. Because of that, the friction will not exert any torque with respect to the point which would lie on the rotational axis, if the bottle were to wobble.

The bottle on the sway will begin to wobble if the torque of the forces acting parallel to the plane of the swing is greater than the torque of forces acting in the axis of the bottle. Let us denote φ an angle of deflection of the swing from vertical. Since Jarda swings harmonically the time dependence of the size of the angle is

$$\varphi = \varphi_0 \sin(\omega t) \;,$$

where φ_0 is the maximum deflection and $\omega = 2\pi f$. Angular velocity and angular acceleration are

$$\dot{\varphi} = \omega\varphi_0 \cos(\omega t) = \omega\sqrt{\varphi_0^2 - \varphi^2}\,,$$
$$\ddot{\varphi} = -\omega^2\varphi_0 \sin(\omega t) = -\omega^2\varphi\,.$$

Let us begin with the gravitational force. We will decompose it into two components with perpendicular directions, analogous to how we would do it in an inclined plane problem. The component of the gravitational force with direction of the normal to the plane of the swing is

$$F_g^{\mathrm{n}} = mg\cos\varphi\,,$$

where m is the mass of the bottle. In the tangential direction to the plane acts component of the gravitational force with magnitude

$$F_g^{\mathrm{t}} = mg\sin\varphi\,.$$

Both components of the gravitational force act in the centre of the mass of the bottle. We calculate the moment arm of torques with respect to the lower edge of the bottle around which the bottle could theoretically rotate. For calculating magnitudes of torques we just need to take the perpendicular distance with respect to the direction of the force to the rotational axis, we get

$$M_g^{\mathrm{n}} = Rmg\cos\varphi\,,$$
$$M_g^{\mathrm{t}} = \frac{h}{2}mg\sin\varphi\,.$$

Both torques are acting against each other.

We will now consider the inertial forces which depend on the distance from the axis of rotation. For the centre of the lower base of the bottle it is

$$l = \sqrt{L^2 - \left(\frac{D}{2}\right)^2} \doteq 1.386\,\mathrm{m}\,.$$

Let us denote x the perpendicular distance from the plane of the swing and let us consider an element of the bottle in the shape of an annulus with a height $\mathrm{d}x$, a with $\mathrm{d}r$ in the distance r from the axis of the cylinder and x from the plane of the swing. We will examine the effect of individual inertial forces on this small volume.

We will begin with the centrifugal force. The angular velocity is the same for all points but the distance from the axis of rotation and the direction of force changes. We will divide the annulus into angular elements $\mathrm{d}\alpha$, where α denotes the angular distance of such an element from a line that is parallel with the axis of swinging and lies in the plane of the chosen annulus. Let us denote β the angle which is formed by a connecting line of an element $\mathrm{d}\alpha$ and the swinging axis with the axis perpendicular to the plane of the swing. The magnitude of centrifugal force is then

$$\mathrm{d}F_\mathrm{o} = \dot{\varphi}^2 \frac{l - x}{\cos\beta}\,\mathrm{d}m\,,$$

$$\mathrm{d}m = \rho\,\mathrm{d}V \ = \rho r\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}\alpha\,.$$

Let us decompose this force into a component that is perpendicular to the plane of the swing, pointing "into" it, and a component that is parallel with the plane. Due to the circular symmetry is the resultant of these parallel components zero. Components acting into the swing are

$$\mathrm{d}F_\mathrm{o}^\mathrm{n} = \mathrm{d}F_\mathrm{o}\cos\beta = \dot{\varphi}^2\,(l-x)\,\rho r\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}\alpha\,.$$

We integrate with respect to the angle α, then to r and finally to x

$$\begin{aligned} F_\mathrm{o} &= \int_0^h \int_0^R \int_0^{2\pi} \dot{\varphi}^2\,(l-x)\,\rho r\,\mathrm{d}\alpha\,\mathrm{d}r\,\mathrm{d}x = \dot{\varphi}^2\rho \int_0^h \int_0^R 2\pi\,(l-x)\,r\,\mathrm{d}r\,\mathrm{d}x = \\ &= 2\pi\dot{\varphi}^2\rho \int_0^h \frac{R^2}{2}\,(l-x)\,\mathrm{d}x = \pi R^2\dot{\varphi}^2\rho\left(lh - \frac{h^2}{2}\right)\,. \end{aligned}$$

We found the magnitude of the centrifugal force, but we are more interested in the torque. Moment arm is $R + r\sin\alpha$, so for the element of torque we get

$$\mathrm{d}M_\mathrm{o} = (R + r\sin\alpha)\,\mathrm{d}F_\mathrm{o}\,.$$

We integrate this expression again over the whole volume of the cylinder. We can notice that right after the integration over the angle α we will get the same expression as what we get when calculating a magnitude of force multiplied by R. The resulting torque is

$$M_\mathrm{o} = RF_\mathrm{o} = \pi R^3\dot{\varphi}^2\rho h\left(l - \frac{h}{2}\right) = mR\dot{\varphi}^2\left(l - \frac{h}{2}\right)\,.$$

We can see that the torque of the centrifugal force is the same as if we replaced all of the mass of the object by a mass point in the centre of mass. This torque returns the bottle into a standing position. We will calculate the torque of inertial force in the same manner. We will come to conclusion that there is inertial force acting upon individual elements of cylinder with height $\mathrm{d}x$

$$\mathrm{d}F_\mathrm{s} = \ddot{\varphi}\,(l - x)\,\mathrm{d}m = \pi\ddot{\varphi}\,(l - x)\,\rho R^2\,\mathrm{d}x\,.$$

These forces also act in the centres of those thin cylinders and their direction is parallel with the plane of the swing. We calculate their moment arm also from the lower edge of the bottle and it has a length x. Torque of inertial force is

$$M_\mathrm{s} = \int_0^h \pi\ddot{\varphi}\,(l - x)\,x\rho R^2\,\mathrm{d}x = \pi\ddot{\varphi}\rho R^2\left(\frac{lh^2}{2} - \frac{h^3}{3}\right) = \ddot{\varphi}mh\left(\frac{l}{2} - \frac{h}{3}\right)\,.$$

This torque has the same direction as the torque of the centrifugal force, it acts against the wobbling. The resulting torque acting upon the bottle is

$$M = M_g^\mathrm{t} - M_\mathrm{s} - M_\mathrm{o} - M_g^\mathrm{n}\,,$$

while if $M > 0$, the bottle will begin to wobble. When we substitute into the equation and simplify, we get

$$M = \frac{h}{2}g\sin\varphi - \frac{h}{2}\omega^2\varphi\left(l - \frac{2}{3}h\right) - Rg\cos\varphi - R\omega^2\left(\varphi_0^2 - \varphi^2\right)\left(l - \frac{h}{2}\right)\,.$$

It is obvious that when $\varphi = 0$ then $M < 0$ and the bottle is stable. Solution of this problem is to find the minimal angle φ_0 for which exists an angle φ such that $|\varphi| < \varphi_0$, while satisfying $M(\varphi, \varphi_0) = 0$.

Instead of verifying all possible combinations of φ and φ_0 we can calculate for all φ_0 in which φ has M maximum and verify if in given φ and φ_0 the torque is positive of negative. For every fixed φ_0 the torque M is a function of variable φ, so for finding the extreme we differentiate

$$\frac{\mathrm{d}M}{\mathrm{d}\varphi} = \frac{h}{2}g\cos\varphi - \frac{h}{2}\omega^2\left(l - \frac{2}{3}h\right) + Rg\sin\varphi + 2R\omega^2\varphi\left(l - \frac{h}{2}\right)\,.$$

Let us put this expression equal to zero and let us denote the solution of the equation φ_m. Notice that φ_m does not depend on φ_0, which means that the maximum will be the same for all φ_0. To be more precise, it would be the same if all of the values of φ were allowed for given φ_0. Since $|\varphi| < \varphi_0$ holds for $|\varphi_0| < |\varphi_\mathrm{m}|$, there is not a point with derivative equal to zero in allowed interval, so we have to look for maximum at the boundary, i.e. in points $\pm\varphi_0$.

The value of φ_m has to be calculated numerically. The only one satisfying $|\varphi_\mathrm{m}| \leq \pi/2$ is $\varphi_\mathrm{m} \doteq -0.781$, but in this point the $M(\varphi = \varphi_0 = \varphi_\mathrm{m})$ is even smaller than M in zero. Therefore it is not a maximum but a minimum. That means that for positive values of φ the torque should be increasing to the next root of the last equation, which is at the point 1.633. Maxima of torques for $0 < \varphi_0 < \pi/2$, therefore, must lie at the boundary which corresponds to the turning point.

Now we need to find the minimal positive φ_0 which satisfies $M(\varphi = \varphi_0) = 0$

$$0 = M(\varphi = \varphi_0) = \frac{h}{2}g\sin\varphi_0 - \frac{h}{2}\omega^2\varphi_0\left(l - \frac{2}{3}h\right) - Rg\cos\varphi_0\,.$$

The result of this straightforward numerical calculation is $\varphi_0 \doteq 0.342$. Notice that for $\varphi = \varphi_0$ the part which contains centrifugal force equals zero because at that moment the angular velocity of the bottle equals zero.

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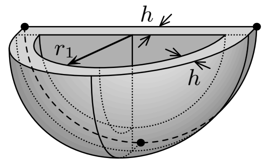
Problem 52 ... washbasin

8 points



A washbasin has two parts: the front part is shaped like a quarter of a sphere, with an inner radius of $r_1 = 30\,\mathrm{cm}$, *and it is attached to the back part in the shape of a half-cylinder. The thickness of all its walls is* $h = 2.0\,\mathrm{cm}$ *and the whole washbasin is made of a material with a density of* $\rho = 3\,100\,\mathrm{kg{\cdot}m^{-3}}$. *It is attached to a wall at two points located at its only two vertices (corners). Other than that, its bottom edge leans freely on the wall. What is the minimum force which the hanging mechanism at each corner must withstand when the basin is completely filled with water? Assume that the bottom edge acts on the wall only at its lowest point and only in the direction perpendicular to the wall.* *Dodo was repairing a bathroom tap.*

Let us first introduce the coordinate system in which we are going to solve the problem. Let its origin be in the middle of the top edge of the closer side to the wall. Let the x-axis be perpendicular to the wall, the y-axis points downwards, and let the z-axis point along the edge of the washbasin. In a static situation, the resultant force must be zero, and also the resultant torque must be zero.

Since the plane xy is the plane of symmetry of the washbasin, a gravitational force of magnitude $F_g = mg$ acts upon the washbasin at its center of gravity with coordinates $(x_T, y_T, 0)$, pointing straight downward. Next, at the lower edge (at the point $(0, r_2, 0)$, where $r_2 = r_1 + h$) a force of unknown magnitude $\mathbf{F}_1 = (F_1, 0, 0)$ acts upon the washbasin in a direction perpendicular to the wall. Finally, at the hinge points (i.e. $(0, 0, \pm r_2)$) a force $\mathbf{F}$ acts upon the washbasin in an unknown direction. Since the magnitude of this force has to be as small as possible, its components must be $\mathbf{F} = (F_x, F_y, 0)$.

The balance of forces gives us the equations

$$F_1 + 2F_x = 0\,,$$
$$F_g + 2F_y = 0\,.$$

From the balance of torques, we have one non-trivial equation.

$$-r_2 F_1 + x_T F_g = 0\,.$$

In total, we have three equations for the three unknowns, while we are looking for the magnitude of a force

$$F = \sqrt{F_x^2 + F_y^2} = \frac{mg}{2}\sqrt{1 + \left(\frac{x_T}{r_2}\right)^2}\,. \tag{4}$$

Therefore we have to calculate the mass of a full washbasin and a x coordinate of its center of gravity. First, let us deal with an object of the shape of a quarter sphere, with a radius R, and homogeneous density distribution. Its volume is apparently

$$V(R) = \frac{\pi}{3}R^3\,,$$

however, determining the position of the centre of gravity is more demanding. If we place the object in the aforementioned coordinate system, from the mirroring the xy plane we have $z_T = 0$, and from the symmetry with respect to the plane $x = y$ we obtain $x_T = y_T$. Thus, the centre of gravity has to be somewhere on this line. The position of a centre of gravity of a homogeneous object is by definition

$$\mathbf{x}_T = \frac{1}{M}\int_V \rho(\mathbf{x})\,\mathbf{x}\,\mathrm{d}V = \frac{1}{V}\int_V \mathbf{x}\,\mathrm{d}V\,.$$

We are interested only in the x coordinate of the centre of gravity which we denote as X_T. We transfer to spherical coordinates (r,θ,φ), where $r\in\langle 0,R\rangle$ is the distance from the origin, $\theta\in\langle 0,\pi\rangle$ is the deflection from the z axis, and $\varphi\in\left\langle 0,\frac{\pi}{2}\right\rangle$ is the angle between the projection of the position vector onto the xy plane, and the x axis. The transformation relation is $x = r\sin\theta\cos\varphi$, the volume differential changes is then $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

From the definition above, we have

$$\begin{aligned}
X_T &= \frac{1}{V}\int_V x\,\mathrm{d}V = \frac{1}{V}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{R} r^3\sin^2\theta\cos\varphi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \\
&= \frac{1}{V}\left[\frac{r^4}{4}\right]_0^R\int_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2\theta\cos\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{R^4}{4V}\left[\sin\varphi\right]_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2\theta\,\mathrm{d}\theta = \\
&= \frac{R^4}{4V}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{R^4}{4V}\left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi} = \frac{\pi R^4}{8V} = \frac{3}{8}R\,.
\end{aligned}$$

Now, we can put the acquired knowledge together. We consider that the washbasin consists of a quarter sphere of radius r_2 with density ρ centered at the point $(h,0,0)$, another quarter sphere of radius r_1 with density $\rho_{\mathrm{v}}-\rho<0$ centered at the same point, and a half cylinder forming the back part of the washbasin of radius r_2 and heigth h. The quantity ρ_{v} denotes the density of water. For the total mass we have

$$m = \frac{\pi}{3}r_2^3\rho + \frac{\pi}{3}r_1^3\left(\rho_v-\rho\right) + \frac{\pi}{2}r_2^2 h\rho \doteq 56.97\,\mathrm{kg}\,.$$

By taking the weighted arithmetic mean for the positon of the centre of gravity in the x direction, we get

$$\begin{aligned}
x_T &= \frac{1}{m}\left[\frac{\pi}{3}r_2^3\rho\left(h+\frac{3}{8}r_2\right) + \frac{\pi}{3}r_1^3\left(\rho_v-\rho\right)\left(h+\frac{3}{8}r_1\right) + \frac{\pi}{2}r_2^2 h\rho\frac{h}{2}\right]\,,\\
x_T &\doteq 12.5\,\mathrm{cm}\,.
\end{aligned}$$

When we put this into the equation (4), we obtain the magnitude of the force $F\doteq 300\,\mathrm{N}$ that each of the hanging mechanisms has to bear.

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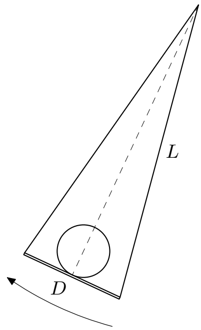
Problem 53 . . . bottle on a swing reloaded

8 points



Jarda is swinging harmoniously on a swing with a frequency $f = 0.5\,\mathrm{Hz}$ *and a maximum inclination angle* 5°*. Next to him, there is an empty, relatively narrow, and long cylinder-shaped bottle, exactly in the middle of the width of the swing. The bottle is placed horizontally, and its axis is perpendicular to the direction of motion. The swing is* $D = 40\,\mathrm{cm}$ *wide and hangs on ropes that are* $L = 3\,\mathrm{m}$ *long. At first, Jarda is holding the bottle, but he releases it when he passes through the equilibrium position. When does the bottle fall off the swing? Assume that the bottle is rolling without slipping.*

Jarda drank a bottle of wine and started thinking about problems.

Let's work in a swing-fixed frame. The forces acting on the bottle are gravity, normal force from the swing seat, inertial force, and centrifugal force. These forces can be decomposed into parallel and perpendicular components with respect to the swing plane.

The Coriolis force acts only perpendicular to the velocity vector. The bottle moves only on the desk, so this force acts only into the desk or in the opposite direction. In this case, the Coriolis force might be greater than the gravity force, but it doesn't occur in this problem (see below).

The centrifugal force is not constant throughout the bottle; however, the distance from the axis to the center of the swing is much greater than the diameter of the bottle, so we can neglect this, and we can assume the bottle to be a point mass in its center of gravity. The resultant of this force points in the "axis of the rotation – the center of gravity of the bottle" line, and is not perpendicular to the pad. It is of magnitude

$$F_\mathrm{c} = m\dot{\varphi}^2 \frac{L}{\cos\alpha}\,,$$

where m is the mass of the bottle, $\dot{\varphi}$ is the angular velocity of the swing, and α is an angle adjacent to the rotation axis, covering the bottle and the center of the swing. The distance between the rotational axis and center of the swing is not exactly L, but regarding the given, this approximation is adequate. Since the bottle is slim, we do not take into account the distance between the center of gravity of the bottle and the center of the swing. Thus, we split the centrifugal force into two components. Since the bottle is moving in the plane, the perpendicular force doesn't affect its motion. The component parallel to the desk is of magnitude

$$F_\mathrm{c}^\mathrm{t} = F_\mathrm{c}\sin\alpha = m\dot{\varphi}^2 L\tan\alpha = m\dot{\varphi}^2 x\,,$$

where x is distance between the bottle and the center of the swing.

The parallel component of the gravity force is

$$F_g^\mathrm{t} = mg\sin\varphi\,,$$

where m is mass of the bottle, $\varphi = \varphi_0\sin(\omega t)$ is the instantaneous angle of the swing, $\varphi_0 = 5^\circ$ is the swing amplitude, $\omega = 2\pi f$ is the angular frequency of swinging and t is time since the release of the bottle. Because the amplitude is reasonably small, we can use the approximation $\sin x \approx x$. The parallel component of the gravity force then becomes $F_g^\mathrm{t} \approx mg\varphi$.

The other component of the gravity force is aproximately mg (because φ_0 is small). Maximal magnitude of Coriolis force is $2\omega m\varphi_0 v \doteq 0.55mv$. The velocity of the bottle must be approximately $18\,\mathrm{m\cdot s^{-1}}$ to make the Coriolis force greater than the gravity force. This doesn't occur in this problem, so the Coriolis force is not important in our solution.

Then there is also the inertial force emerging from the acceleration and deceleration of the swing. Due to the negligible radius of the bottle compared to the rope length L, the acceleration caused by inertial force is

$$a_\mathrm{s} \approx \ddot{\varphi} = -\varphi_0\omega^2 \sin(\omega t)\, L = -\omega^2 L\varphi\,.$$

The inertial force magnitude is $F_\mathrm{s} = ma_\mathrm{s}$, and it points opposite to the tangential acceleration of the swing, i.e., opposite to the parallel component of the gravity force, which we calculated in the previous paragraph. The magnitude of the parallel component of inertial force is independent of the position of the bottle relative to the swing. When the bottle moves off the swing axis, the inertial force changes both magnitude and direction; nevertheless, its parallel component magnitude stays constant.

Now we can see the magnitude of the tangential component of the centrifugal force F_c^t is negligible to the inertial and gravity force because there is a small angle φ_0 squared, but in the inertial and gravity force, it occurs only in the first power.

The resultant force acting on the bottle in direction prallel to the swing plane is

$$F = F_g^\mathrm{t} - F_\mathrm{s} = m\varphi\left(g - \omega^2 L\right)\,.$$

Notice that if the swing oscillated at an angular frequency of a mathematical pendulum, the angular frequency would be $\omega^2 = \frac{g}{L}$ and the two forces would exactly cancel out each other. However, the swing oscillates with frequency $f = 0.5\,\mathrm{Hz}$ which is not a frequency of mathematical pendulum with length L.

We can approximate that the force acts in the center of the bottle (because the length of the rope is large compared to the bottle radius). The torque with respect to the point of contact between the bottle and swing is $M = RF$, where R is bottle radius. The moment of inertia of the bottle for rotation around this point is from the parallel axis theorem

$$J = J_\mathrm{s} + mR^2 = mR^2 + mR^2 = 2mR^2\,,$$

where $J_\mathrm{s} = mR^2$ is the moment of inertia of a cylindrical shell (we neglect the bottom and the neck of the bottle). Let x be the distance of the bottle center from the swing axis. We derive the equation

$$\ddot{x} = \varepsilon R = \frac{MR}{J} = \frac{\varphi\left(g - \omega^2 L\right)}{2} = \frac{\varphi_0\left(g - \omega^2 L\right)}{2}\sin(\omega t)\,,$$

where ε is the angular acceleration of the bottle. We integrate the equation and obtain

$$\dot{x} = -\frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega}\cos(\omega t) + C\,.$$

Boundary condition is $\dot{x}(0) = 0$, thus

$$C = \frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega} \quad\Rightarrow\quad \dot{x} = \frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega}\left(1 - \cos(\omega t)\right)\,.$$

By integrating the equation for speed $\dot{x}$, we get the position dependence on time

$$x = \frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega}\left(t - \frac{1}{\omega}\sin(\omega t)\right) = \frac{\varphi_0}{2}\left(\frac{g}{\omega^2} - L\right)(\omega t - \sin(\omega t)) \; .$$

where we used the boundary condition $x(0) = 0$. The bottle will fall off when the position $|x|$ is larger than $\frac{D}{2}$. The time can be calculated from equation

$$D = \left|\varphi_0\left(\frac{g}{\omega^2} - L\right)(\omega t - \sin(\omega t))\right| \; .$$

This equation has to be solved numerically. The bottle will fall at time $t \doteq 0.86\,\mathrm{s}$.

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Problem 54 ... minimisation of acceleration 7 points

We are driving a car, which we consider to be a point mass, at a speed of $v_0 = 20.0\,\mathrm{m{\cdot}s^{-1}}$. At some point, we notice that we are approaching a curve, at the end of which we'll need to stop at traffic lights. Due to our insurance company, we want the maximum absolute value of the acceleration acting on the car (until it stops at the traffic lights) to be as small as possible. If we start braking $d = 30.0\,\mathrm{m}$ before the beginning of the curve, the radius of the curve is $R = 10.0\,\mathrm{m}$ and we turn 22.5° on the curve, what time does it take to reach the end of the curve from the point where we start braking? *Karel's insurance company insists on minimum acceleration.*

Let us denote the acceleration of braking as a. At first, we will drive the distance d as uniformly decelerated motion. By expressing the time from the equation for distance of such a motion, we get

$$d = v_0 t_1 - \frac{1}{2} a t_1^2 \, ,$$
$$t_1 = \frac{v_0 - \sqrt{v_0^2 - 2ad}}{a} \, ,$$

where we do not consider the solution with a positive sign (+) before the square root as it represents the situation when the car does not stop accelerating even after it stops – it would return back. Hence, the car will move at speed $v_1 = v_0 - a t_1 = \sqrt{v_0^2 - 2ad}$ at the beginning of the curve.

Now we get to the point where the situation becomes more complex because the car will also turn in addition to braking. It means that centripetal acceleration (which is always perpendicular to the braking) will act on the car, too. We can obtain the magnitude of centripetal acceleration by expression $a_\mathrm{c} = v^2/R$. Then, if in the sum, we do not want to exceed the acceleration a, it remains $a_\mathrm{b} = \sqrt{a^2 - v^4/R^2}$ on braking. If we denote the speed in time t as $v(t)$, we get the following differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \sqrt{a^2 - v^4/R^2} \, .$$

This is separable; nevertheless, we would get hypergeometric functions from the integral. Furthermore, we need to determine the distance that car would travel somehow to find out whether it would stop before the traffic lights. So if we do not have a smarter idea of how to avoid it, we have to solve the problem numerically.

The outline of the algorithm: We choose the acceleration and compute the car's speed at the beginning of the curve, and the time it needs to get there analytically. Then we proceed by tiny steps and compute the change in distance (and add it to total distance, which it traveled in the curve), and by how much it slows down (and subtract it from the speed) in each of the steps. When speed becomes less than 0 (the car stops), we check the distance that the car already traveled and compare it to the length of the curve, which is $\pi R/8$. If it traveled more, it means we need to increase the acceleration and vice versa. Then we repeat the process with such adjusted acceleration.

Regarding adjusting the acceleration, we can use the bisection method, where the initial interval could be from $a_{\min} = v_0^2/(2d + R)$ (at this acceleration, the car gets to the beginning of the curve at speed, when the centripetal acceleration is equal to $a_{\min}$, thus, for any less acceleration it could not follow the curve) to $a_{\max} = v_0^2/(2d)$ (at this acceleration, the car stops exactly at the beginning of the curve). Hence, the sought acceleration has to be in this interval, and by selecting it, we avoided all potential square roots of negative numbers. When do we stop bisecting the interval? The question in the problem assignment asks about the time duration of the braking. Therefore, in addition to acceleration, we have to remember the particular time at the boundaries of the interval. Then we will bisect until the difference of these 2 times becomes less than the accuracy required by the task.

```
import numpy as np
v0 = 20
d = 30
R = 10
l = np.pi * R / 8
amin = v0 * v0 / (2 * d + R)
amax = v0 * v0 / (2 * d)
tmin = 0.
tmax = 10000.
dt = 0.0001
while abs(tmin - tmax) > 10 * dt:
   a = (amin + amax) / 2
   v = np.sqrt(v0 * v0 - 2 * a * d)
   t1 = (v0 - v) / a
   s = 0.0
   i = 0
   while v > 0:
     i += 1
     s += v * dt
     at = np.sqrt(a * a - v * v * v * v / R / R)
     v -= at * dt
   if s < l:
     amax = a
     tmin = t1 + i * dt
   else:
```

```
        amin = a
        tmax = t1 + i * dt
print(tmin)
print(tmax)
print(a)
```

After the convergence of the script, it will tell us that the optimal acceleration is approximately $6\,\mathrm{m\cdot s^{-2}}$ with the braking time of $3.42\,\mathrm{s}$.

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Problem M.1 ... the boat 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$ and water flows at speed $v_\mathrm{F} = 0.50\,\mathrm{m\cdot s^{-1}}$.

What is the fastest time in which Lego can cross the river if he does not care how far along the shore the current takes him? *Lego just got carried away.*

If we do not mind, how far will the current take us; we shall paddle perpendicular to the river. The perpendicular (to the river) velocity component is then v_B (while the velocity component in the flow direction is v_F). Therefore, Lego will cross the river in time

$$t = \frac{l}{v_\mathrm{B}} = 5.0\,\mathrm{s}\,.$$

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Problem M.2 ... the boat reloaded 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$ and water flows at speed $v_\mathrm{F} = 0.50\,\mathrm{m\cdot s^{-1}}$ in the whole river.

What is the fastest time in which Lego can cross the river if he wants to move in the direction perpendicular to the water flow at all times?

Lego found out that the shortest path is not always the fastest one.

To move perpendicular to the river at any time, we need the paddle velocity component in the flow direction to cancel out with the velocity of the river flow i.e. the magnitude of this component has to be equal to v_F. Therefore; the component perpendicular to the river will be

$$v_\mathrm{n} = \sqrt{v_\mathrm{B}^2 - v_\mathrm{F}^2} \doteq 0.87\,\mathrm{m\cdot s^{-1}}\,.$$

Thus, we will cross the river in time

$$t = \frac{l}{v_\mathrm{n}} \doteq 5.8\,\mathrm{s}\,.$$

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Problem M.3 ... the boat revolutions 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m{\cdot}s^{-1}}$. He wants to move in the direction perpendicular to the river again; however, it has been raining and the water seems to flow faster. What is the slowest speed of the flow for which he cannot cross the river?

Legolas needed one more problem for the series.

In the previous problem, we found out that if we want to move perpendicular to the river, the magnitude of the final velocity will be

$$v_\mathrm{n} = \sqrt{v_\mathrm{B}^2 - v_\mathrm{F}^2}\,,$$

where v_F is the speed of the water flow. This equation returns positive final speed for all $v_\mathrm{F} < v_\mathrm{B}$; therefore for the speed of water flow slower than our maximum speed, Lego can cross the river.

On the other hand, the equation for $v_\mathrm{F} > v_\mathrm{B}$ does not make sense; therefore for such velocities, Lego is not able to cross the river. In the case when $v_\mathrm{F} = v_\mathrm{B}$, the final velocity is zero i.e. Lego will paddle at the same spot - the river will not take him away; however, he will not be moving forward either.

Thus, for the case when $v_\mathrm{F} = v_\mathrm{B} = 1.0\,\mathrm{m{\cdot}s^{-1}}$, Lego already cannot cross the river.

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Problem M.4 ... the boat resurrections 4 points

Lego wants to cross a river of a width $l = 6.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m{\cdot}s^{-1}}$. The water flows at a speed $v_\mathrm{F} = 1.5\,\mathrm{m{\cdot}s^{-1}}$. From the previous problem, Lego already knows that in this situation, it is impossible to move in the direction perpendicular to the river. He would like to know the minimum distance by which the current would take him away (from the point on the other shore of the river directly opposite to his starting point) while crossing the river. *Lego optimizes.*

One way to find the solution is to express the distance (that the current will take him away) for a general direction of paddling and then find the minimum (e.g. by derivative).

Another way to find the solution is by geometric thought. To express the questioned distance, we need only the direction of the final velocity. This distance will be minimum when the angle between the direction of the final velocity and the direction of the river flow will be maximum.

This occurs when the direction of the final velocity is perpendicular to the direction of the river flow (the triangle consisting of the velocity of the river flow, the paddling velocity, and the final velocity will be a right triangle, while the hypotenuse will be the river flow velocity).

The maximum angle between the final velocity and the riverbank can be computed as

$$\varphi = \arcsin\frac{v_\mathrm{B}}{v_\mathrm{F}} \doteq 42^\circ\,.$$

The distance that the current will take Lego away is

$$d = \frac{l}{\tan\varphi} \doteq 6.7\,\mathrm{m}\,.$$

Note that the width of the river is greater (by one meter) than in previous problems.

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Problem E.1 ... discharging a capacitor 3 points

We are discharging a capacitor of a capacitance $C = 42.0\,\mathrm{mF}$ through a resistor with a resistance of $R = 9.81\,\mathrm{k\Omega}$. We know that the equation $u(t) = U_0 \mathrm{e}^{-\frac{t}{RC}}$ holds for the evolution of voltage in time. In what time will the capacitor's electric field have half the energy of the fully charged state? *Karel wanted the participants to get familiar with discharging.*

The energy of a capacitor is

$$E = \frac{1}{2} C U^2 \,.$$

Thus, the problem can be described by the following equation

$$\frac{1}{4} C U_0^2 = \frac{1}{2} C U_0^2 \mathrm{e}^{-2\frac{t}{RC}} \quad \Rightarrow \quad \frac{1}{2} = \mathrm{e}^{-2\frac{t}{RC}} \,,$$

which we solve for t.

Now, we have two options how to get the solution. We can substitute the equation directly into some computational software, which will solve it for us. An example is Wolfram Alpha.[5] By this approach, we directly get the solution $t \doteq 143\,\mathrm{s}$.

If we are familiar with the logarithmic functions, we can choose an analytical path that is not so demanding, too. We take advantage of the fact that $x = \mathrm{e}^{\ln x}$, and adjust the equation to

$$\frac{1}{2} = \mathrm{e}^{\ln\frac{1}{2}} = \mathrm{e}^{-\ln 2} \quad \Rightarrow \quad -\ln 2 = -2\frac{t}{RC} \,,$$

$$t = \frac{RC}{2} \ln 2 \doteq 143\,\mathrm{s} \,.$$

The energy on the capacitor drops by half in 143 s.

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Problem E.2 ... conductive chewing gum 4 points

Rado is chewing a chewing gum with a specific electrical resistance $\rho = 1\,\mathrm{m\Omega{\cdot}mm}$ and volume $V = 1\,\mathrm{cm^3}$. He splits it in half and uses each half as a separate conductor. In his backpack, he has an electrical appliance of an internal resistance $R = 1\,\Omega$ that can only be powered by voltages up to $U_{\max} = 5\,\mathrm{V}$, and a DC voltage source $U = 12\,\mathrm{V}$. What is the minimum distance l between the electrical appliance and the power supply if he wants to connect them with the two conductors made of chewing gum? The conductors should have the same constant cross-sections. *Rado had chewing gum after a long time.*

The volume of both gums is

$$V_{\mathrm{gum}} = \frac{V}{2} = Sl \,,$$

[5] https://www.wolframalpha.com/input/?i=exp%5B-2*t%2F%289.81*10%5E3*42*10%5E%28-3%29%29%5D+%3D+1%2F2

where S is the area of the cylinder base and l is its length (the distance between the appliance and the source). The resistance of one chewing gum is calculated by the specific electrical resistance as follows

$$R_{\mathrm{gum}} = \rho \frac{l}{S} = \rho \frac{l^2}{V_{\mathrm{gum}}} = 2\rho \frac{l^2}{V}\,.$$

The total resistance of the circuit is $R_{\mathrm{c}} = 2R_{\mathrm{gum}} + R$. From Ohm's law, we obtain

$$U = R_{\mathrm{c}} I = (2R_{\mathrm{gum}} + R) I = \left(4\rho \frac{l^2}{V} + R\right) I\,,$$

where $I = \frac{U_{\max}}{R}$, and thus

$$l = \sqrt{\frac{VR\left(\frac{U}{U_{\max}} - 1\right)}{4\rho}} \doteq 0.59\,\mathrm{m}\,.$$

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Problem E.3 ... cheap voltmeter 3 points

An AC voltmeter works as follows. At first, the DC component is filtered out of the input voltage using a high-pass filter. Subsequently, the obtained voltage is rectified and smoothed by a low-pass filter. The resulting DC voltage is measured and converted to the displayed value by multiplying it by a numerical factor. This factor is chosen in such a way that for a sine wave, the displayed value is the effective value of voltage. What value is being displayed if the input AC voltage has a sawtooth waveform oscillating between $U_1 = 0.00\,\mathrm{V}$ and $U_2 = 1.00\,\mathrm{V}$?

Dodo steals ideas for problems from the Department of Physics Education.

Firstly, we determine the numerical factor used by the device. After the rectification of harmonic voltage given by the relation $U = U_0 \sin(\omega t)$, which does not have a DC component (its time mean is zero), we get the timecourse of the voltage $U = U_0 \left|\sin(\omega t)\right|$.

Subsequent smoothing makes a mean value of it. The rectified signal is π-periodic, so we only need to calculate the mean of the following expression

$$U_{\mathrm{m}} = \frac{1}{\pi} \int_0^{\pi} U_0 \sin\varphi \,\mathrm{d}\varphi = \frac{2}{\pi} U_0\,.$$

To obtain the RMS value given by the mean square of the voltage, we have to multiply the measured voltage by the conversion factor a.

$$U_{\mathrm{eff}} = \sqrt{\frac{1}{\pi} \int_0^{\pi} U_0^2 \sin^2\varphi \,\mathrm{d}\varphi} = \frac{1}{\sqrt{2}} U_0\,.$$

The conversion factor therefore has a value of

$$a = \frac{U_{\mathrm{eff}}}{U_{\mathrm{m}}} = \frac{\pi}{2\sqrt{2}}\,.$$

In the case of a sawtooth waveform voltage, we have to firstly remove the DC component – the long-term time mean is equal to $U = 0.50\,\mathrm{V}$, so the resulting sawtooth waveform has values between $-0.50\,\mathrm{V}$ and $0.50\,\mathrm{V}$. After the rectification, we get consecutive triangles all with a base on the timeline and a height of $0.50\,\mathrm{V}$.

After smoothing, we get the mean value of voltage $U'_\mathrm{m} = 0.25\,\mathrm{V}$ and after conversion in the device to "effective value", the displayed value is $U_\mathrm{d} = aU'_\mathrm{m} \doteq 0.28\,\mathrm{V}$. By the way, the actual value of the effective value is $U'_\mathrm{eff} = \frac{1}{\sqrt{3}}\,\mathrm{V} \doteq 0.58\,\mathrm{V}$ i.e. almost twice as high.

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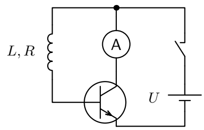
Problem E.4 ... coil on a base — 4 points

Consider a circuit with a DC source ($U > 50\,\mathrm{V}$), an inductor with inductance $L = 5\,\mathrm{H}$ and resistance $R = 10\,\mathrm{k\Omega}$, an NPN transistor, an ideal ammeter, and a switch. Some time after closing the switch, the value on the ammeter will settle on $I = 1\,\mathrm{A}$. At what time after closing the switch does the ammeter display the value $I' = 500\,\mathrm{mA}$?



Vojta was not building a bomb.

Notice, that we can omit the BE voltage thanks to $U > 50\,\mathrm{V}$. Let β be the current gain of the transistor. We can express the currents flowing through the inductor in terms of currents flowing through the ammeter. In the time ammeter was displaying value I, the current $I_\mathrm{max} = I/\beta$ was flowing through the inductor, and analogously when ammeter was showing I', $I_L = I'/\beta$ flowed through the inductor.

Now we proceed by using the following relation, which can be easily derived by solving a differential equation for self-induction of a coil,

$$I_L = I_{max}\left(1 - e^{-\frac{R}{L}t}\right),$$

from where it is straightforward to substitute for the currents and rearrange the terms to get

$$t = \frac{L}{R}\ln\left(\frac{I}{I - I'}\right) \doteq 0.35\,\mathrm{ms}\,.$$

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Problem X.1 ... I'll pull her down with me — 3 points

Let us assume we have a ball of air of radius $R = 2.1\,\mathrm{cm}$, which is wrapped in a layer of polystyrene of thickness $t = 2.1\,\mathrm{cm}$ and density $\rho = 33\,\mathrm{kg{\cdot}m^{-3}}$. We attach the ball to a force gauge by a rigid, weightless string and then pull it down so that half of the ball's volume is under the water surface. What is the value we see on the force gauge?

Karel had balloons on his mind.

The force gauge will display the difference of the buoyancy force (that will keep the ball floating) and the gravity acting on the ball (where we neglect the mass of the air). The volume of the polystyrene is the difference of the volume of the whole ball V_{ball} and the volume of the air V_{a}. Let us denote the density of the water as ρ_{w}. Then

$$F = F_{\text{b}} - F_g = \frac{1}{2} V_{\text{ball}}\, \rho_{\text{w}} g - (V_{\text{ball}} - V_{\text{a}})\, \rho g$$
$$= \frac{2}{3}\pi\, (R+t)^3\, \rho_{\text{w}} g - \left(\frac{4}{3}\pi\, (R+t)^3 - \frac{4}{3}\pi R^3\right) \rho g \doteq 1.43\,\mathrm{N}\,.$$

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Problem X.2 ... to the bottom of the dam — 4 points

Consider a piece of polystyrene with weight $m_0 = 1.25\,\mathrm{kg}$ and density $\rho_0 = 27\,\mathrm{kg{\cdot}m^{-3}}$ that we want to completely submerge to the bottom of a water dam. We have plenty of stones of various sizes and the same density $\rho = 2\,650\,\mathrm{kg{\cdot}m^{-3}}$. What is the minimum weight of a stone which we should attach to the polystyrene? *Karel was thinking about Archimedes.*

Let's denote the weight of the stone by m. After submerging the polystyrene-stone system in water, it is subjected to buoyancy $F_{\text{b}} = V\rho g$ and gravity $F = Mg$ force, where $V = \frac{m_0}{\rho_0} + \frac{m}{\rho}$ is the total volume and $M = m_0 + m$ is the total weight.

In the extreme case, the magnitude of the gravitational force will be the same as the magnitude of the buoyancy force

$$(m_0 + m)g = \left(\frac{m_0}{\rho_0} + \frac{m}{\rho}\right) \rho_{\text{w}} g\,,$$

where $\rho_{\text{w}} = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water.

By solving this equation we get

$$m = m_0 \frac{\frac{\rho_{\text{w}}}{\rho_0} - 1}{1 - \frac{\rho_{\text{w}}}{\rho}} \doteq 72.1\,\mathrm{kg}\,.$$

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Problem X.3 ... an unknown ball in an unknown fluid — 4 points

We are given a ball of unknown radius, submerged in a fluid of unknown (but constant) density. We know that the difference between hydrostatic pressures at the top and the bottom point of the ball is $\Delta p = 850\,\mathrm{Pa}$, and the total buoyant force acting on the ball is $F_{\text{b}} = 150\,\mathrm{N}$. What is the density of the fluid? *Lego was inventing buoyancy problems.*

The hydrostatic pressure difference can be expressed as $\Delta p = \Delta h \rho g$. In this problem, the height difference between the top and the bottom point of the ball is the diameter of the ball i.e. double the radius ($\Delta h = 2r$). Therefore, we can express the radius of the ball as

$$r = \frac{\Delta p}{2\rho g}\,.$$

Because the volume of the ball is $V = 4/3\pi r^3$, we only need to substitute into the formula for buoyancy force

$$F_\mathrm{b} = V\rho g = \frac{4}{3}\pi\frac{\Delta p^3}{8\rho^3 g^3}\rho g\,,$$

now, we can express the density of the unknown fluid

$$\rho = \sqrt{\frac{\pi\Delta p^3}{6F_\mathrm{b}g^2}} \doteq 150\,\mathrm{kg\cdot m^{-3}}\,.$$

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Problem X.4 ... attractive

3 points

Danka has two brass balls with diameters of 2.00 cm *carrying the same charges. One of the balls is fixed and the other one hovers above it at a distance of* $d_1 = 20.0\,\mathrm{cm}$ *(we assume that the ball can only move in the vertical direction). We now immerse both of the balls in oil of density* $\rho_0 = 910\,\mathrm{kg\cdot m^{-3}}$*, where a new equilibrium is established. The balls are now* $d_2 = 14.0\,\mathrm{cm}$ *away from each other. What is the relative permittivity of the oil we used? Consider the density of brass to be* $\rho = 8\,400\,\mathrm{kg\cdot m^{-3}}$*. Distance between the two balls means the distance of their centers.*

Danka was playing with balls.

To solve the problem, we need to examine the forces acting on the upper ball. In the air, it is subjected to the electrostatic force given by Coulomb's law which compensates for the gravitational force. Therefore, we have

$$V\rho g = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{d_1^2}\,,$$

where $V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$ is the volume of one ball, ε_0 is the permittivity of vacuum and Q is the magnitude of electric charge of one ball. When immersed in oil, in addition to these two forces, a significant hydrostatic buoyancy force acts on the upper ball. This situation has the equilibrium described by a similar relation

$$V\rho g = \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}}\frac{Q^2}{d_2^2} + V\rho_0 g\,,$$

where ε_r is the relative permittivity of oil, we're searching for. The first and the second equilibrium relations give us two equations of two unknowns. To simplify the calculation, we express the fraction from the first relation as

$$\frac{Q^2}{4\pi\varepsilon_0}\,,$$

and plug it into the second one. From here, we can express the relative permittivity of oil as

$$\varepsilon_\mathrm{r} = \left(\frac{d_1}{d_2}\right)^2\frac{\rho}{\rho - \rho_0}\,.$$

After substituting with numerical values, we get that the relative permittivity of the oil we used is $\varepsilon_\mathrm{r} \doteq 2.3$.

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